HOMEWORK PROBLEMS III

Due Tuesday April 2nd -
6.1.8 For the following toric varieties X_Σ, compute $\mathrm{Pic}(X_\Sigma)$ and describe which torus-invariant divisors are basepoint free.
(a) X_Σ is the toric variety of the smooth complete fan Σ in $\mathbb{R}^2$ with

$$\Sigma(1) = \{\pm e_1, \pm e_2, e_1 + e_2\}.$$

(b) The blowup $\mathrm{Bl}_p(\mathbb{P}^{)}$ of $\mathbb{P}^n$ at a fixed point p of the torus action.

6.1.9 The fan of $(\mathbb{P}^1)^n$ has ray generators $\pm e_1, \ldots, \pm e_n$. Let $D_1^\pm, \ldots, D_n^\pm$ denote the corresponding torus-invariant divisors. Consider $D = \sum_{i=1}^n (a_i^+ D_i^+ + a_i^- D_i^-)$. When is D basepoint free?

Due Thursday April 4th -
6.1.8 + 6.1.9(cont'd) For the toric varieties in the previous homework, describe which torus-invariant divisors are ample.
6.1.3 In class we saw that if D is ample on a complete toric variety, then P_D is a full dimensional lattice polytope. Here you will show that the same statement is false if *ample* is replaced by *basepoint free*. Consider $(\mathbb{P}^1)^n$ and for any $0 < d < n$ find a basepoint free divisor on $(\mathbb{P}^1)^n$ such that $\dim P_D = d$.

Due Tuesday April 9th -
6.3.5 Consider the complete fan in $\mathbb{R}^3$ with six minimal generators

$$u_1 = (1,0,1), u_2 = (0,1,1), u_3 = (-1,-1,1)$$
$$u_4 = (1,0,-1), u_5 = (0,1,-1), u_6 = (-1,-1,-1)$$

and six maximal cones

$$\mathrm{Cone}(u_1,u_2,u_3), \mathrm{Cone}(u_1,u_2,u_4), \mathrm{Cone}(u_2,u_4,u_5)$$
$$\mathrm{Cone}(u_1,u_3,u_4,u_6), \mathrm{Cone}(u_2,u_3,u_5,u_6), \mathrm{Cone}(u_4,u_5,u_6).$$

(a) Draw a picture of this fan.

(b) Prove that $\mathrm{Pic}(X_\Sigma) \cong \{a(D_1 + D_4) | a \in 3\mathbb{Z}\}$ and that the nef cone in $N^1(X_\Sigma) \cong \mathrm{Pic}(X_\Sigma)_\mathbb{R}$ is $\{a(D_1 + D_4) | a \geq 0\}$.

(c) Prove that X_Σ is not projective.

Note that $D = 3(D_1 + D_4)$ is in the interior of the nef cone, but is not ample.

Due Thursday April 11th -
6.4.6 Let X_Σ be the blowup of $\mathbb{P}^n$ at a fixed point of the torus action. Thus $\mathrm{Pic}X_\Sigma \cong \mathbb{Z}^2$.
(a) Compute the nef and Mori cones of X_Σ and draw pictures of them like we did for the Hirzebruch surface in class.
(b) Determine the extremal walls.

Due Tuesday April 16th -
None!

Due Tuesday April 23rd -
In this exercise we recall Cox's construction of a toric variety X_Σ with no torus factors as a good categorical quotient and then use this description to relate graded modules for the total coordinate ring with quasicoherent sheaves on X_Σ.

Let $S = \mathbb{C}[x_\rho | \rho \in \Sigma(1)]$ be the total coordinate ring of X_Σ. We defined its irrelevant ideal to be $B(\Sigma) = \{x^{\hat{\sigma}} | \sigma \in \Sigma\}$. We then defined $Z(\Sigma) = \mathbb{V}(B(\Sigma)) \subset \mathbb{C}^{\Sigma(1)}$ and considered $\mathbb{C}^{\Sigma(1)} \setminus Z(\Sigma)$. Let G be the kernel of the natural map of tori $(\mathbb{C}^*)^{\Sigma(1)} \to T_N$. We saw that G has character group $\mathrm{Cl}(X_\Sigma)$ and acts on $\mathbb{C}^{\Sigma(1)} \setminus Z(\Sigma)$.

Considering $\mathbb{C}^{\Sigma(1)} \setminus Z(\Sigma)$ as a toric variety itself, we constructed the obvious map of fans to give a map $\pi : \mathbb{C}^{\Sigma(1)} \setminus Z(\Sigma) \to X_\Sigma$.

Finally we showed that π is a good categorical quotient by G, by checking that for each $\sigma \in \Sigma$, the preimage of U_σ is $U_{\tilde{\sigma}}$ where $\tilde{\sigma} = \mathrm{Cone}(e_\rho | \rho \in \sigma(1))$, and the map $\pi : U_{\tilde{\sigma}} \to U_\sigma$ is a good categorical quotient. This was in turn equivalent to showing the map $\pi_\sigma^* : \chi^m \mapsto x^{\langle m \rangle} = \prod_\rho x_\rho^{\langle m, u_\rho \rangle}$ induces as isomorphism

$$\pi_\sigma^* : \mathbb{C}[\sigma^\vee \cap M] \xrightarrow{\sim} (S_{x^{\hat{\sigma}}})^G \subset S_{x^{\hat{\sigma}}}.$$

1. Show that $S_{x^{\hat{\sigma}}}$ is graded by $\mathrm{Cl}(X_\Sigma)$. $(S_{x^{\hat{\sigma}}})_0 = (S_{x^{\hat{\sigma}}})^G$.

2. Let M be a graded S-module (graded by the class group). Show that there is a quasicoherent sheaf $\tilde{M}$ on X_Σ such that for every $\sigma \in \Sigma$,

$$\Gamma(U_\sigma, \tilde{M}) = (M_{x^{\tilde{\sigma}}})_0.$$

3. For any $\alpha \in \mathrm{Cl}(\mathrm{X}_\Sigma)$. Let $S(\alpha)$ be the graded free S-module such that $S(\alpha)_\beta = S_{\alpha+\beta}$. By part (2), this gives a quasicoherent (in fact, coherent) sheaf $\mathcal{O}_{X_\Sigma}(\alpha)$. Show

$$S_\alpha \cong \Gamma(X_\Sigma, \mathcal{O}_{X_\Sigma}(\alpha)).$$

4. Let $D = \sum_\rho a_\rho D_\rho$ be a Weil divisor such that $[D] = \alpha$. Prove

$$\mathcal{O}_{X_\Sigma} \cong \mathcal{O}_{X_\Sigma}(\alpha).$$