A SUM-PRODUCT THEOREM IN SEMI-SIMPLE COMMUTATIVE BANACH ALGEBRAS

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0. Introduction

Let A be a finite subset of $\mathbb{R}$. It was proven by Erdös and Szemerédi [E-S] that the sumset $A+A=\{x+y : x,y\in A\}$ and product set $A\cdot A=\{x\cdot y : x,y\in A\}$ cannot be both 'small'. More precisely, they showed that $|A+A|+|A\cdot A|>c_1|A|^{1+C}$ for some constant $c>0$ and they conjectured that $|A+A|+|A\cdot A|>c_\varepsilon|A|^{2-\varepsilon}$ for all $\varepsilon>0$. This problem is still open and the best result to date due to Solymosi [Sol], stating that

$$|A+A|+|A\cdot A|>|A|^{\frac{14}{11}-\varepsilon} \tag{0.1}$$

Part of the interest nowadays in this type of questions comes from its relevance to certain issues in Analysis centered around the dimension conjectures for 'Kakeya sets' in $\mathbb{R}^d$ $(d\geq 3)$ and related problems (see [K-T], [T], [Bo] for more details on the matter). Most of them are far from solved but methods from 'arithmetic combinatorics' permitted to make certain progress. Naturally, this circle of ideas has a counterpart in the finite field setting, replacing $\mathbb{R}$ by $\mathbb{F}_q$. If q is prime, a sum-product theorem of the Erdös-Szemerédi type was obtained in [B-K-T], based on an argument due to Edgar and Miller in their solution of the Erdös-Volkmann ring problem (see [E-M]). Besides the applications in [B-K-T], that result turned out to be an interesting application to Gauss-sum estimates over prime fields when the degree is large (see [B-K]). It is shown in [B-K] that given $\varepsilon>0$, there is $\delta>0$ such that for p prime and $k<p^{1-\varepsilon}$, one has

$$\max_{a\neq 0(p)}\left|\sum_{x=0}^{p-1} e^{\frac{2\pi i}{p}ax^k}\right| < cp^{1-\delta}. \tag{0.2}$$

Sum-product problems for sets of complex numbers were considered in [Ch1], [Ch2], [Ch3] and [E]. We will consider here a setting which is significantly different, in the sense that zero-divisor problems do appear.

Theorem 1. *There is a constant $\nu > 0$ such that if A is a finite set of a semi-simple commutative Banach algebra R, then*

$$|A + A| + |A \cdot A| > c|A|^{1+\nu}. \tag{0.3}$$

Since $\mathcal{R}$ admits a faithful representation as a function space on the regular maximal ideal space $\mathfrak{M}$ (the Gelfand representation),it is semi-simple. Theorem 1 is obviously equivalent to the following more elementary statement.

Theorem 2. *Let A be a finite subset of the infinite product-algebra $\prod \mathbb{R}$ or $\prod \mathbb{C}$ with coordinate-wise addition and multiplication. Then (0.3) holds, for some absolute constant $\nu > 0$.*

We don't know the optimal exponent ν. However, and this is perhaps the most interesting point, examples show that ν may *not* be taken arbitrarily close to 1. In fact

Remark 0.4. Theorem 2 does not hold for $\nu > 1 - \frac{\log 2}{\log 3}$.

This is seen as follows. Let $A = \{1, \cdots, M\} \times \{0, 1\}^m \subset \mathbb{R} \times \mathbb{R}^m$, thus $|A| = N = M2^m$. Since

$$A + A \subset \{1, \dots, 2M\} \times \{0, 1, 2\}^m$$
$$A \cdot A \subset \{1, \dots, M^2\} \times \{0, 1\}^m$$

it follows that $|A + A| \leq 2M3^m$ and $|A \cdot A| \leq M^2 2^m$.

Taking $M \sim (\frac{3}{2})^m$ gives the desired conclusion.

As mentioned, the issue of zero-divisors is a significant problem (although not the only one). Notice that in case of bounded dimension, thus $A \subset \mathbb{R}^t$ with t fixed, this problem is easily avoided. Indeed, there is a subset $A' \subset A, |A'| \geq 2^{-t}|A|$ such that for each $i = 1, \dots, t$, the coordinate projection $\pi_i(A')$ is either $\{0\}$ (in which case the i-coordinate may be ignored) or $\pi_i(A') \subset \mathbb{R}\backslash\{0\}$.

An important point when treating the general case, is the 'dimensional reduction' based on the smallness of the sumset. Freiman's lemma implies indeed that if $A \subset \prod \mathbb{R}, |A| < \infty$ satisfies $|A + A| \leq t|A|$, then there is a subset I of the index set, $|I| \leq t$, such that the coordinate projection $\pi_I : \prod \mathbb{R} \to \prod_I \mathbb{R}$ is one-to-one when restricted to A, It is therefore no surprise that the size of the additive doubling constant $\frac{|A+A|}{|A|}$ does play a significant role in the combinatorics. Our main technical lemma in this respect is Lemma 3.1 below, which is the base of the multi-scale analysis (this lemma is very similar to certain constructions in [B-C] but the context here is different).

Finally, notice that the assumption of semi-simplicity is obviously necessary. Theorem 1 clearly fails for $R = \{ \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} : x \in \mathbb{C}\}$.

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1. Sum-Product for Graphs on $\mathbb{R}$

Proposition 1.1. *Let $S \subset \mathbb{R}$ be a finite set, $|S| = N$ and $\mathcal{G} \subset S \times S$ with*

$$|\mathcal{G}| \geq \delta N^2.$$

Then

$$|S \underset{\mathcal{G}}{+} S| \cdot |S \underset{\mathcal{G}}{\times} S| > c\delta^4 N^{5/2}. \tag{1.1}$$

Proof. We use Elekes' method.

Consider the points

$$\{(x+z, yz) : (x,z) \in \mathcal{G}, (y,z) \in \mathcal{G}\} \subset (S \underset{\mathcal{G}}{+} S) \times (S \underset{\mathcal{G}}{\times} S).$$

Let $n \in \mathbb{Z}_+$ to be specified. From Szemerédi-Trotter

$$|S \underset{\mathcal{G}}{+} S|^2 |S \underset{\mathcal{G}}{\times} S|^2 > cn^3 |\{(x,y) \in S \times S : |\mathcal{G}_x \cap \mathcal{G}_y| \sim n\}|. \tag{1.2}$$

Our aim is to make the right side of (1.2) large.

We have by Cauchy-Schwartz

$$\delta N^2 \leq \sum_{x \in S} |\mathcal{G}_x| = \sum_{z \in S} \sum_{x \in S} \chi_{\mathcal{G}_x}(z) \leq N^{1/2} \Bigg[\sum_{z \in S} \Big(\sum_{x \in S} \chi_{\mathcal{G}_x}(z) \Big)^2 \Bigg]^{1/2}$$

$$\leq N^{1/2} \Big(\sum_{x,y \in S} |\mathcal{G}_x \cap \mathcal{G}_y| \Big)^{1/2},$$

hence

$$\sum_{x,y \in S} |\mathcal{G}_x \cap \mathcal{G}_y| > \delta^2 N^3. \tag{1.3}$$

Since $|\mathcal{G}_x \cap \mathcal{G}_y| \leq N$, (1.3) implies that for some $n \in \mathbb{Z}_+$

$$n \cdot |\{(x,y) \in S \times S : \; |\mathcal{G}_x \cap \mathcal{G}_y| \sim n\}| > \frac{\delta^2 N^3}{\log \frac{1}{\delta}}. \tag{1.4}$$

From (1.4), we have in particular

$$n > \frac{\delta^2 N}{\log \frac{1}{\delta}}. \tag{1.5}$$

Substituting (1.4) and (1.5) in (1.2), we have

$$|S \underset{\mathcal{G}}{+} S|^2 \cdot |S \underset{\mathcal{G}}{\times} S|^2 > \frac{\delta^6 N^5}{(\log \frac{1}{\delta})^3}$$

which implies (1.1).

Remark 1.1.1. Proposition 1.1 fails in dimension 2. If $A \subset \mathbb{R}$ is a finite set, then $S \subset \mathbb{R} \times \mathbb{R}$ as $S = (A \times \{0\}) \cup (\{0\} \times A)$. Let $\mathcal{G} \subset S \times S$ be the graph

$$\mathcal{G} = \{\big((x,0),(0,y)\big) : x, y \in A\}.$$

Then

$$S \underset{\mathcal{G}}{+} S = A \times A \text{ and } S \underset{\mathcal{G}}{\times} S = \{(0,0)\}.$$

Thus

$$|S \underset{\mathcal{G}}{+} S| \cdot |S \underset{\mathcal{G}}{\times} S| = N^2.$$

2. Addition constant and multiplication constant.

Let

$$\mathcal{R} = \prod_{j=1}^{t} \mathbb{R}.$$

Let $A_1, A_2 \subset \mathcal{R}$ be finite sets

$$|A_i| = N_i$$

and $\mathcal{G} \subset A_1 \times A_2$

$$|\mathcal{G}| = \delta N_1 N_2, \quad 0 < \delta < 1.$$

We define the sum and product sets of A_1, A_2 along the graph $\mathcal{G}$

$$A_1 \underset{\mathcal{G}}{+} A_2 = \{x + y = (x_j + y_j)_j : (x,y) \in \mathcal{G}\}$$

$$A_1 \underset{\mathcal{G}}{\times} A_2 = \{x \cdot y = (x_j y_j)_j : (x,y) \in \mathcal{G}\},$$

and addition and multiplication constants

$$K_+(\mathcal{G}) = \frac{|A_1 \underset{\mathcal{G}}{+} A_2|}{\sqrt{N_1 N_2}} \tag{2.1}$$

$$K_\times(\mathcal{G}) = \frac{|A_1 \underset{\mathcal{G}}{\times} A_2|}{\sqrt{N_1 N_2}}. \tag{2.2}$$

Thus

$$\frac{\delta \max(N_1, N_2)}{\sqrt{N_1 N_2}} \leq K_+(\mathcal{G}) \leq \delta\sqrt{N_1 N_2} \tag{2.3}$$

and

$$\frac{1}{\sqrt{N_1 N_2}} \leq K_\times(\mathcal{G}) \leq \delta\sqrt{N_1 N_2}.$$

Lemma 2.1. *If $\mathcal{G} \subset A_1 \times A_2, A_i \subset \mathbb{R}$, then*

$$K_+(\mathcal{G})^{1-\theta} \cdot K_\times(\mathcal{G})^\theta > \delta^2 (N_1 N_2)^{\frac{\theta}{4}} \text{ for all } 0 \leq \theta \leq \frac{2}{15}.$$

Proof.

Let $S = A_1 \cup A_2 \subset \mathbb{R}$ and consider $\mathcal{G} \subset A_1 \times A_2 \subset S \times S$.

Assume $N_1 \geq N_2$. Hence $N = |S| \sim N_1$ and $\frac{|\mathcal{G}|}{N^2} > \delta \cdot \frac{N_2}{N_1}$.

From (1.1)

$$K_+ \cdot K_\times N_1 N_2 = |A_1 \underset{\mathcal{G}}{+} A_2| \cdot |A_1 \underset{\mathcal{G}}{\times} A_2| > C\delta^4 \left(\frac{N_2}{N_1}\right)^4 N_1^{5/2} > c\delta^4 N_1^{-3/2} N_2^4$$

$$K_+ \cdot K_\times > c\delta^4 N_1^{-5/2} N_2^3. \tag{2.4}$$

Also from (2.3)

$$\delta N_1 \leq K_+ \sqrt{N_1 N_2} \Rightarrow N_2 > \left(\frac{\delta}{K_+}\right)^2 N_1. \tag{2.5}$$

From (2.4), (2.5)

$$K_+ \cdot K_\times > c\delta^4 (N_1 N_2)^{1/4} \left(\frac{N_2}{N_1}\right)^{11/4} > c\delta^4 \left(\frac{\delta}{K_+}\right)^{11/2} (N_1 N_2)^{1/4}$$

$$K_+^{\frac{13}{2}} \cdot K_\times > c\delta^{\frac{19}{2}} (N_1 N_2)^{1/4}$$

$$K_+^{1-\frac{2}{15}} K_\times^{\frac{2}{15}} > c\delta^{\frac{19}{15}} (N_1 N_2)^{1/30}. \tag{2.6}$$

Also

$$K_+ \geq \delta \tag{2.7}$$

and (2.4) follows from interpolation between (2.6), (2.7).

3. Factorization Lemma

Fix $0 < \theta < \frac{2}{15}$.

Define

$$\beta(N, \delta, K) = \beta_\theta(N, \delta, K) = \min K_+(\mathcal{G})^{1-\theta} K_\times(\mathcal{G})^\theta N^{\frac{1}{2}}. \tag{3.1}$$

where the minimum is taken over all $A_1, A_2 \subset \mathcal{R}, \mathcal{G} \subset A_1 \times A_2$ such that

$$|A_i| = N_i, \quad \text{for } i = 1, 2 \tag{3.2}$$

$$N = N_1 N_2 \tag{3.3}$$

$$|\mathcal{G}| \geq \delta N \tag{3.4}$$

$$K_+(\mathcal{G}) < K.$$

Lemma 3.1.

$$\beta(N, \delta, K) > \min \frac{\delta^{11}}{(\log \frac{K}{\delta})^{38}} \beta(N', \delta', K') \beta(N'', \delta'', K'') \left(\frac{N}{N'N''} \right)^{\frac{1}{2} + \frac{\theta}{4}} \tag{3.5}$$

where the minimum is taken over

$$N', N'' < \left(\frac{K}{\delta} \right)^{15} N^{1/2} \tag{3.6}$$

$$N'N'' < N$$

$$\delta' \cdot \delta'' > (\log \frac{K}{\delta})^{-4} \delta$$

$$K' \cdot K'' < \delta^{-6} (\log \frac{K}{\delta})^4 K.$$

Proof.

For $i = 1, 2$, let $A_i \subset \mathcal{R}$ and $\mathcal{G} \subset A_1 \times A_2$ satisfy (3.2)-(3.4). For each i, we want to find a subset of A_i with "regular" structure, i.e. the sizes of the fibers over points in the subset, of certain coordinate projection, have the same magnitude.

First, we want to construct $A_i' \subset A_i$ with

$$|A_i'| > \frac{3\delta}{4}|A_i| \tag{3.7}$$

such that for any $B_i \subset A_i'$,

$$|\mathcal{G} \cap (B_1 \times A_2')| > \frac{\delta}{4}|B_1||A_2'| \tag{3.8}$$

$$|\mathcal{G} \cap (A_1' \times B_2)| > \frac{\delta}{4}|A_1'||B_2| \tag{3.9}$$

and

$$(\mathcal{G} \cap (A_1' \times A_2')^c| \leq \frac{\delta}{4}|(A_1 \times A_2)\backslash(A_1' \times A_2')|. \tag{3.10}$$

It is clear that (3.10) implies (3.7). Indeed,

$$|A_1'||A_2'| \geq |\mathcal{G} \cap (A_1' \times A_2')| > \delta N_1 N_2 - \frac{\delta}{4}N_1 N_2 = \frac{3\delta}{4}N_1 N_2.$$

We obtain A_i' by removing any bad subset B_i. Assume $|\mathcal{G} \cap (B_1 \times A_2')| \leq \frac{\delta}{4}|B_1|\ |A_2'|$ for some $B_1 \subset A_1'$. Let $A_1'' = A_1' \backslash B_1$. We see that $A_1'' \times A_2'$ satisfies (3.10).

$$\begin{aligned} |\mathcal{G} \cap (A_1'' \times A_2')^c| &= |\mathcal{G} \cap (A_1' \times A_2')^c| + |\mathcal{G} \cap (B_1 \times A_2')| \\ &\leq \frac{\delta}{4}|(A_1 \times A_2)\backslash(A_1' \times A_2')| + \frac{\delta}{4}|B_1|\ |A_2'| \\ &= \frac{\delta}{4}|(A_1 \times A_2)\backslash(A_1'' \times A_2')|. \end{aligned}$$

Continuing removing the bad set B_i, (3.10) ensures that the remaining set is still big enough, and the process gives the desired result.

Next, we want to split $\mathcal{R} = \prod_{j=1}^t \mathbb{R}$ into two parts.

For $1 \leq j \leq t$, consider the decreasing functions for $i = 1, 2$,

$$n_i(j) = \max_{(x_1,\dots,x_j)\in\mathbb{R}^j} |A_i(x_1, \dots, x_j)|,$$

where $A_i(x_1, \dots, x_j) = \{(x_{j+1}, \dots, x_t) \mid (x_1, \dots, x_t) \in A_i\}$ is the fiber of A_i over the point $(x_1, \cdots, x_j)$.

We take t' such that

$$\begin{cases} n_1(t') + n_2(t') \geq N^{1/4} \\ n_1(t'+1) + n_2(t'+1) \leq N^{1/4}. \end{cases}$$

We assume $n_1(t') \geq n_2(t')$, thus

$$n_1(t') \geq \frac{1}{2} N^{1/4}. \tag{3.11}$$

Let $\mathcal{R}_1 = \prod_{j=1}^{t'} \mathbb{R}$, and $\mathcal{R}_2 = \prod_{j=t'+1}^{t} \mathbb{R}$, and let $\pi_1 : \mathcal{R} \to \mathcal{R}_1$ be the projection to the first t' coordinates.

Denote

$$\bar{x} = (x_1, \cdots, x_{t'}).$$

In what follows, denote $K_+(\mathcal{G})$ by K.

Claim 1. *There exists a set* $\bar{\bar{A}}_2 \subset A_2'$ *with* $|\bar{\bar{A}}_2| > c\frac{\delta^3}{\log \frac{K}{\delta}} N_2$*, such that for all* $\bar{x} \in \pi_1(\bar{\bar{A}}_2)$*, we have* $|\bar{\bar{A}}_2(\bar{x})| \sim m_2 > c\delta^5 K^{-2} N^{1/4}$*, and* $|\pi_1(\bar{\bar{A}}_2)| < C\delta^{-5} K^2 \frac{N_2}{N^{1/4}}$.

Proof. Let $\bar{x} \in \pi_1(A_1')$ such that

$$|A_1'(\bar{x})| = n_1(t').$$

It follows from (3.8) that

$$|\mathcal{G} \cap [(\{\bar{x}\} \times A_1'(\bar{x})) \times A_2']| > \frac{\delta}{4} n_1(t') |A_2'|$$

and hence there is a subset $A_2'' \subset A_2'$ such that, by the Fact stated at the end of this proof,

$$|A_2''| > \frac{\delta}{8} |A_2'| > \frac{3\delta^2}{32} N_2, \tag{3.12}$$

and for $z \in A_2''$

$$|\mathcal{G} \cap [(\{\bar{x}\} \times A_1'(\bar{x})) \times \{z\}]) > \frac{\delta}{8} n_1(t'). \tag{3.13}$$

From (2.5) and (3.13), we get clearly

$$\begin{aligned} \frac{K^2}{\delta} N_2 \geq K\sqrt{N_1 N_2} &= |A_1 \underset{\mathcal{G}}{+} A_2| \\ &\geq |(\{\bar{x}\} \times A_1'(\bar{x})) \underset{\mathcal{G}}{+} A_2''| \\ &> \frac{\delta}{8} |\pi_1(A_2'')| \cdot n_1(t'). \end{aligned} \tag{3.14}$$

Let $\bar{A}_2 \subset A_2''$ such that the fibers over any $\bar{x} \in \pi_1(\bar{A}_2)$ have size at least $\frac{\delta^5 n_1(t')}{10^4 K^2}$, i.e.

$$\bar{A}_2 = \bigcup_{|A_2''(\bar{x})| > 10^{-4}\delta^5 K^{-2} n_1(t')} \{\bar{x}\} \times A_2''(\bar{x}).$$

It follows from (3.14) that

$$|A_2'' \backslash \bar{A}_2| \le |\pi_1(A_2'')| 10^{-4}\delta^5 K^{-2} n_1(t') < \delta^3 10^{-3} N_2 < \frac{\delta}{10}|A_2''| \tag{3.15}$$

The last inequality is by (3.7) and (3.12).

Since by (3.9)

$$|\mathcal{G} \cap (A_1' \times A_2'')| > \frac{\delta}{4}|A_1'|\,|A_2''|,$$

it follows from (3.15) that

$$|\mathcal{G} \cap (A_1' \times \bar{A}_2)| > \frac{\delta}{4}|A_1'|\,|A_2''| - \frac{\delta}{10}|A_1'|\,|A_2''| > \frac{\delta}{10}|A_1'|\,|A_2''|.$$

Since $|A_2''(\bar{x})| \le n_2(t') \le n_1(t')$, we may specify m_2 and $\bar{\bar{A}}_2$ as follows:

$$10^{-4}\delta^5 K^{-2} n_1(t') < m_2 < n_1(t'), \tag{3.16}$$

and

$$A_2' \supset A_2'' \supset \bar{A}_2 \supset \bar{\bar{A}}_2 = \bigcup_{|A_2''(\bar{x})| \sim m_2} \big(\{\bar{x}\} \times A_2''(\bar{x})\big)$$

such that

$$|\mathcal{G} \cap (A_1' \times \bar{\bar{A}}_2)| > c\frac{\delta}{\log \frac{K}{\delta}}|A_1'|\,|A_2''|. \tag{3.17}$$

Thus $\bar{\bar{N}}_2 := |\bar{\bar{A}}_2|$ satisfies

$$\bar{\bar{N}}_2 := |\bar{\bar{A}}_2| > c\frac{\delta}{\log \frac{K}{\delta}}|A_2''| > c\frac{\delta^3}{\log \frac{K}{\delta}} N_2. \tag{3.18}$$

By (3.16) and (3.11)

$$|\bar{\bar{A}}_2(\bar{x})| \sim m_2 > c\delta^5 K^{-2} N^{1/4}, \tag{3.19}$$

and

$$|\pi_1(\bar{\bar{A}}_2)| \sim \frac{|\bar{\bar{A}}_2|}{m_2} < \frac{|A_2|}{m_2} < C\delta^{-5} K^2 \frac{N_2}{N^{1/4}}. \tag{3.20}$$

Fact. *Let* $|E| \le e$ *and* $|F| \le f$. *If* $|\mathcal{G} \cap (E \times F)| > \alpha e f$, *then there exists* $F' \subset F$ *with* $|F'| > \frac{\alpha}{2} f$, *such that for any* $z \in F'$, $|\mathcal{G} \cap (E \times \{z\})| > \frac{\alpha}{2} e$.

Now we are ready to find subset of A_1' with regular structure.

Claim 2. *There exists a set* $\bar{\bar{A}}_1 \subset A_1'$ *with* $|\bar{\bar{A}}_1| > c\frac{\delta^2}{(\log \frac{K}{\delta})^2} N_1$, *such that for any* $\bar{x} \in \pi_1(\bar{\bar{A}}_1)$, *we have* $|\bar{\bar{A}}_1(\bar{x})| \sim m_1 > c\delta^{10} K^{-5} N^{1/4}$, $|\pi_1(\bar{\bar{A}}_1)| < C\delta^{-10} K^5 \frac{N_1}{N^{1/4}}$, *and* $|\,\mathcal{G} \cap (\bar{\bar{A}}_1 \times \bar{\bar{A}}_2)| > c\frac{\delta}{(\log \frac{K}{\delta})^2} |\bar{\bar{A}}_1|\,|\bar{\bar{A}}_2|$.

Proof. We observe that for any $\tilde{A}_1 \subset A_1'$, if

$$|\mathcal{G} \cap (\tilde{A}_1 \times \bar{\bar{A}}_2)| \sim |\mathcal{G} \cap (A_1' \times \bar{\bar{A}}_2)|, \tag{3.21}$$

then

$$m := \max_{\bar{x} \in \pi_1(\tilde{A}_1)} |\tilde{A}_1(\bar{x})| \, > \, c\frac{\delta^4}{(\log \frac{K}{\delta})^2} K^{-2} m_2. \tag{3.22}$$

Indeed, from (3.21), (3.17) and the regular structure of $\bar{\bar{A}}_2$, there is $\bar{x} \in \pi_1(\bar{\bar{A}}_2)$ such that

$$|\mathcal{G} \cap \big(\tilde{A}_1 \times \big(\{\bar{x}\} \times \bar{\bar{A}}_2(\bar{x})\big)\big)| > c\frac{\delta}{\log \frac{K}{\delta}}\, |A_1'|\, m_2.$$

Hence by the Fact above, there is a subset $A_1'' \subset \tilde{A}_1 \subset A_1'$ satisfying

$$|A_1''| > c\frac{\delta}{\log \frac{K}{\delta}}\, |A_1'| \tag{3.23}$$

and for any $z \in A_1''$

$$|\mathcal{G} \cap (\{z\} \times (\{\bar{x}\} \times \bar{\bar{A}}_2(\bar{x})))| > c\frac{\delta}{\log \frac{K}{\delta}} m_2.$$

Same reasoning as in (3.14), we have

$$\begin{aligned} \frac{K^2}{\delta} N_1 \ge K\sqrt{N_1 N_2} \ge |A_1 \underset{\mathcal{G}}{+} A_2| &\ge |A_1'' \underset{\mathcal{G}}{+} \big(\{\bar{x}\} \times \bar{\bar{A}}_2(\bar{x})\big)| \\ &> c|\pi_1(A_1'')| \frac{\delta}{\log \frac{K}{\delta}} m_2 \\ &> c\frac{|A_1''|}{m}\; \frac{\delta}{\log \frac{K}{\delta}} m_2 \\ &> c\frac{\delta^3}{(\log \frac{K}{\delta})^2}\; \frac{m_2}{m} N_1. \end{aligned}$$

The last two inequalities are by the definition of m in (3.22) and (3.23), (3.7). Hence

$$m > c\frac{\delta^4}{(\log\frac{K}{\delta})^2}K^{-2}m_2.$$

Since the bound in (3.22) is bigger than $\delta^5 K^{-3} m_2$. Therefore, in (3.17) we may replace A_1' by $\bar{A}_1$ defined as follows.

$$A_1' \supset \bar{A}_1 = \bigcup_{|A_1'(\bar{x})| > \delta^5 K^{-3} m_2} \big(\{\bar{x}\} \times A_1'(\bar{x})\big).$$

Thus, applying (3.22) to $A_1' - \bar{A}_1$, we see that

$$|\mathcal{G} \cap (\bar{A}_1 \times \bar{\bar{A}}_2)| > c\frac{\delta}{\log\frac{K}{\delta}}|A_1'||A_2''|.$$

Recalling (3.16), for $\bar{x} \in \pi_1(\bar{A}_1)$

$$\delta^5 K^{-3} m_2 < |\bar{A}_1(\bar{x})| \le n_1(t') < C\delta^{-5}K^2 m_2.$$

Keeping (3.17) and (3.21) in mind, we may thus again specify

$$\delta^5 K^{-3} m_2 < m_1 < C\delta^{-5} K^2 m_2 \tag{3.24}$$

such that the regular set $\bar{\bar{A}}_1$ defined as

$$A_1' \supset \bar{A}_1 \supset \bar{\bar{A}}_1 = \bigcup_{|\bar{A}_1(\bar{x})| \sim m_1} \big(\{\bar{x}\} \times \bar{A}_1(\bar{x})\big)$$

will satisfy

$$|\mathcal{G} \cap (\bar{\bar{A}}_1 \times \bar{\bar{A}}_2)| > c\frac{\delta}{(\log\frac{K}{\delta})^2}|A_1'|\,|A_2''|. \tag{3.25}$$

Now, (3.25), (3.7) and the fact that $\bar{\bar{A}}_i \subset A_i''$ give

$$\bar{\bar{N}}_1 := |\bar{\bar{A}}_1| > c\frac{\delta^2}{(\log\frac{K}{\delta})^2}N_1 \tag{3.26}$$

and

$$|\,\mathcal{G} \cap (\bar{\bar{A}}_1 \times \bar{\bar{A}}_2)| > c\frac{\delta}{(\log\frac{K}{\delta})^2}\bar{\bar{N}}_1\bar{\bar{N}}_2. \tag{3.27}$$

It follows from (3.20) and (3.24) that

$$m_1 > c\delta^{10}K^{-5}N^{1/4},$$
$$|\pi_1(\bar{\bar{A}}_1)| \sim \frac{|\bar{\bar{A}}_1|}{m_1} < \frac{|A_1|}{m_1} < C\delta^{-10}K^5\frac{N_1}{N^{1/4}}. \tag{3.28}$$

Now, we will give regular structure to the graph $\mathcal{G}$.

Notation. For simplicity, we denote $\bar{\bar{A}}_1, \bar{\bar{A}}_2$ by A_1, A_2 with cardinalities $\bar{\bar{N}}_i$ satisfying (3.18) and (3.26).

Claim 3. *There exists a graph* $\mathcal{G}_{1,1} \subset \pi_1(A_1) \times \pi_1(A_2) \subset \mathcal{R}_1 \times \mathcal{R}_1$ *with* $|\,\mathcal{G}_{1,1}| > \delta_0 |\pi_1(A_1)|\,|\pi_1(A_2)|$ *, such that* $\forall (\bar{x}_1, \bar{x}_2) \in \mathcal{G}_{1,1}$*, we have* $|A_1(\bar{x}_1) \underset{\mathcal{G}_{\bar{x}_1,\bar{x}_2}}{+} A_2(\bar{x}_2)| \sim L\sqrt{m_1 m_2}$*, with* $L < L_0$*, and* $|\,\mathcal{G}_{\bar{x}_1,\bar{x}_2}| \sim \delta_1 m_1 m_2$*, where* $\mathcal{G}_{\bar{x}_1,\bar{x}_2}$ *is the fiber of* $\mathcal{G}$ *over* $(\bar{x}_1, \bar{x}_2)$*, and* δ_0, δ_1 *and* L_0 *satisfy (3.33), (3.29) and (3.49) respectively.*

For $\bar{x}_1, \bar{x}_2 \in \mathcal{R}_1$, let $\mathcal{G}_{\bar{x}_1,\bar{x}_2}$ be the fiber of $\mathcal{G}$ over $(\bar{x}_1, \bar{x}_2)$,

$$\mathcal{G}_{\bar{x}_1,\bar{x}_2} = \{(\bar{y}_1, \bar{y}_2) \in A_1(\bar{x}_1) \times A_2(\bar{x}_2) | \big((\bar{x}_1, \bar{y}_1), (\bar{x}_2, \bar{y}_2)\big) \in \mathcal{G}\} \subset \mathcal{R}_2 \times \mathcal{R}_2.$$

Proof. It follows from (3.27) that we may restrict $\mathcal{G}$ to $\mathcal{G}_1 \times (\mathcal{R}_2 \times \mathcal{R}_2)$, where

$$\mathcal{G}_1 = \{(\bar{x}_1, \bar{x}_2) \in \pi_1(A_1) \times \pi_1(A_2) |\ |\mathcal{G}_{\bar{x}_1,\bar{x}_2}| > c\frac{\delta}{(\log \frac{K}{\delta})^2} m_1 m_2\}.$$

Thus

$$\sum_{(\bar{x}_1,\bar{x}_2) \notin \mathcal{G}_1} |\mathcal{G}_{\bar{x}_1,\bar{x}_2}| \leq c\frac{\delta}{(\log \frac{K}{\delta})^2} \bar{\bar{N}}_1 \bar{\bar{N}}_2,$$

and

$$c\, m_1 m_2 \geq |\mathcal{G}_{\bar{x}_1,\bar{x}_2}| > c\frac{\delta}{(\log \frac{K}{\delta})^2} m_1 m_2, \ \text{ for } (\bar{x}_1, \bar{x}_2) \in \mathcal{G}_1.$$

By (3.27),

$$\sum_{(\bar{x}_1,\bar{x}_2) \in \mathcal{G}_1} |\mathcal{G}_{\bar{x}_1,\bar{x}_2}| > c\frac{\delta}{(\log \frac{K}{\delta})^2} \bar{\bar{N}}_1 \bar{\bar{N}}_2.$$

Also, we may thus specify δ_1,

$$1 > \delta_1 > c\frac{\delta}{(\log \frac{K}{\delta})^2} \tag{3.29}$$

such that if

$$\mathcal{G}_1' = \{(\bar{x}_1, \bar{x}_2) \in \mathcal{G}_1 |\ |\mathcal{G}_{\bar{x}_1,\bar{x}_2}| \sim \delta_1 m_1 m_2\},$$

then we have

$$\sum_{(\bar{x}_1,\bar{x}_2) \in \mathcal{G}_1'} |\mathcal{G}_{\bar{x}_1,\bar{x}_2}| > c\frac{\delta}{(\log \frac{K}{\delta})^3} \bar{\bar{N}}_1 \bar{\bar{N}}_2.$$

(Clearly, $\log \frac{(\log \frac{K}{\delta})^2}{\delta} < \log \frac{K}{\delta}$.)

Hence

$$| \mathcal{G}_1'| > c\frac{\delta}{\delta_1(\log \frac{K}{\delta})^3}\ |\pi_1(A_1)|\ |\pi_1(A_2)|, \tag{3.30}$$

which is bigger than $\frac{\delta}{(\log \frac{K}{\delta})^3}|\pi_1(A_1)|\ |\pi_1(A_2)|$.

By further restriction of $\mathcal{G}_1'$, we will also make a specification on the size of the sumset of $\mathcal{G}_{\bar{x}_1,\bar{x}_2}$.

For $(\bar{x}_1, \bar{x}_2) \in \mathcal{G}_1'$, let $K_+(\mathcal{G}_{\bar{x}_1,\bar{x}_2})$ be the addition constant of $A_1(\bar{x}_1)$ and $A_2(\bar{x}_2)$ along the graph $\mathcal{G}_{\bar{x}_1,\bar{x}_2}$ as defined in (2.1).

First, we see that if $\mathcal{H} \subset \mathcal{G}_1'$, with

$$|\mathcal{H}| \sim |\mathcal{G}_1'| > \frac{\delta}{(\log \frac{K}{\delta})^3}\ |\pi_1(A_1)|\ |\pi_1(A_2)|,$$

then

$$\min_{(\bar{x}_1,\bar{x}_2)\in\mathcal{H}} K_+(\mathcal{G}_{\bar{x}_1,\bar{x}_2}) < L_0 := c^{-1}(\log \frac{K}{\delta})^{\frac{9}{2}}\delta^{-\frac{9}{2}}K. \tag{3.31}$$

In fact, assume for all $(\bar{x}_1, \bar{x}_2) \in \mathcal{H}$ that $K_+(\mathcal{G}_{\bar{x}_1,\bar{x}_2}) > L_0$. Then

$$\begin{aligned} K\sqrt{N_1N_2} \geq |A_1 \underset{\mathcal{G}}{+} A_2| &> \min_{(\bar{x}_1,\bar{x}_2)\in\mathcal{H}}\{|A_1(\bar{x}_1) \underset{\mathcal{G}_{\bar{x}_1,\bar{x}_2}}{+} A_2(\bar{x}_2)|\}|\pi_1(A_1) \underset{\mathcal{H}}{+} \pi_1(A_2)| \\ &\geq L_0\sqrt{m_1m_2}\ \frac{|\mathcal{H}|}{\sqrt{|\pi_1(A_1)|\ |\pi_1(A_2)|}} \\ &> L_0\frac{\delta}{(\log \frac{K}{\delta})^3}\ (\bar{\bar{N}}_1\bar{\bar{N}}_2)^{1/2} \\ &> \delta^{-1}\sqrt{N_1N_2}K, \end{aligned}$$

which is a contradiction. (The last inequality is by (3.18), (3.26) and (3.49).)

Hence, we may reduce $\mathcal{G}_1'$ to $\mathcal{G}_1'' \subset \mathcal{G}_1'$, with $|\mathcal{G}_1''| \sim |\mathcal{G}_1'|$ such that

$$|A_1(\bar{x}_1)\underset{\mathcal{G}_{\bar{x}_1,\bar{x}_2}}{+} A_2(\bar{x}_2)| < L_0\sqrt{m_1m_2} \text{ for } (\bar{x}_1, \bar{x}_2) \in \mathcal{G}_1''.$$

Therefore there is $\mathcal{G}_{1,1} \subset \mathcal{G}_1''$ and $1 < L < L_0$ (see (3.49))

$$|\mathcal{G}_{1,1}| > \frac{c\ |\mathcal{G}_1''|}{\log \frac{K}{\delta}} > \delta_0|\pi_1(A_1)|\ |\pi_1(A_2)|, \tag{3.32}$$

where, by (3.30)

$$\delta_0 > c\,\frac{\delta}{\delta_1(\log\frac{K}{\delta})^4} \tag{3.33}$$

and

$$|A_1(\bar{x}_1) \underset{\mathcal{G}_{\bar{x}_1,\bar{x}_2}}{+} A_2(\bar{x}_2)| \sim L\sqrt{m_1 m_2} \tag{3.34}$$

for $(\bar{x}_1, \bar{x}_2) \in \mathcal{G}_{1,1}$.

Since

$$\begin{aligned} K\sqrt{N_1 N_2} &\geq |\pi_1(A_1) \underset{\mathcal{G}_{1,1}}{+} \pi_1(A_2)||A_1(\bar{x}_1) \underset{\mathcal{G}_{\bar{x}_1,\bar{x}_2}}{+} A_2(\bar{x}_2)| \\ &\geq |\pi_1(A_1) \underset{\mathcal{G}_{1,1}}{+} \pi_1(A_2)| \cdot L\sqrt{m_1 m_2} \\ &= K_+(\mathcal{G}_{1,1}) L \sqrt{\bar{\bar{N}}_1 \bar{\bar{N}}_2}, \end{aligned}$$

we have

$$K_+(\mathcal{G}_{1,1}) \cdot L < \delta^{-\frac{5}{2}} (\log\frac{K}{\delta})^{\frac{3}{2}} K < \delta^{-3} (\log K)^2 K. \tag{3.35}$$

In summary, $\mathcal{G}_{1,1} \subset \pi_1(A_1) \times \pi_1(A_2)$ satisfies (3.32), (3.33) and for $(\bar{x}_1, \bar{x}_2) \in \mathcal{G}_{1,1}$, the graph $\mathcal{G}_{\bar{x}_1,\bar{x}_2} \subset A_1(\bar{x}_1) \times A_2(\bar{x}_2)$ satisfies

$$\begin{aligned} &\{(\bar{x}_1, \bar{x}_2)\} \times \mathcal{G}_{\bar{x}_1,\bar{x}_2} \subset \mathcal{G} \\ &|\mathcal{G}_{\bar{x}_1,\bar{x}_2}| \sim \delta_1 m_1 m_2, \end{aligned} \tag{3.36}$$

where δ_1 is as in (3.29). The addition constants $K_+(\mathcal{G}_{1,1})$ and L satisfy (3.31) and (3.35).

Denote

$$\mathcal{G} \supset \tilde{\mathcal{G}} = \bigcup_{(\bar{x}_1,\bar{x}_2)\in\mathcal{G}_{1,1}} (\{(\bar{x}_1, \bar{x}_2)\} \times \mathcal{G}_{\bar{x}_1,\bar{x}_2}) \tag{3.37}$$

which satisfies

$$|\tilde{\mathcal{G}}| > c\,\frac{\delta}{(\log\frac{K}{\delta})^4}\ \bar{\bar{N}}_1 \bar{\bar{N}}_2 \tag{3.38}$$

where

$$\bar{\bar{N}}_1 \bar{\bar{N}}_2 > \frac{\delta^5}{(\log\frac{K}{\delta})^3} N_1 N_2. \tag{3.39}$$

Next, we will estimate β (see (3.1) for the definition).

From (3.37)

$$A_1 \underset{\mathcal{G}}{+} A_2 \supset A_1 \underset{\tilde{\mathcal{G}}}{+} A_2 = \bigcup_{(\bar{x}_1,\bar{x}_2)\in\mathcal{G}_{1,1}} \big[\{\bar{x}_1+\bar{x}_2\} \times \big(A_1(\bar{x}_1) \underset{\mathcal{G}_{\bar{x}_1,\bar{x}_2}}{+} A_2(\bar{x}_2)\big)\big].$$

Let $M_i = |\pi_1(A_i)|$. Then

$$\begin{aligned} |A_1 \underset{\mathcal{G}}{+} A_2| &\geq K_+(\mathcal{G}_{1,1})\sqrt{M_1M_2} \cdot \min_{(\bar{x}_1,\bar{x}_2)\in\mathcal{G}_{1,1}} |A_1(\bar{x}_1) \underset{\mathcal{G}_{\bar{x}_1,\bar{x}_2}}{+} A_2(\bar{x}_2)| \\ &\geq K_+(\mathcal{G}_{1,1})\sqrt{M_1M_2}\ L\ \sqrt{m_1m_2} \end{aligned} \tag{3.40}$$

by (3.34).

Similarly

$$|A_1 \underset{\mathcal{G}}{\times} A_2| \geq K_\times(\mathcal{G}_{1,1})\sqrt{M_1M_2} \cdot \min_{(\bar{x}_1,\bar{x}_2)\in\mathcal{G}_{1,1}} K_\times(\mathcal{G}_{\bar{x}_1,\bar{x}_2})\sqrt{m_1m_2} \tag{3.41}$$

(notice that we did not regularize with respect to the product).

If we take some $(\bar{x}_1,\bar{x}_2)\in\mathcal{G}_{1,1}$ realizing the minimum in (3.41), it follows from (3.34)

$$\begin{aligned} L^{1-\theta}K_\times(\mathcal{G}_{\bar{x}_1,\bar{x}_2})^\theta\sqrt{m_1m_2} &\sim K_+(\mathcal{G}_{\bar{x}_1,\bar{x}_2})^{1-\theta}K_\times(\mathcal{G}_{\bar{x}_1,\bar{x}_2})^\theta\sqrt{m_1m_2} \\ &\geq \beta(m_1m_2,\delta_1,L) \end{aligned}$$

by definition (3.1) of β and (3.36).

Hence (3.40) and (3.41) give

$$\begin{aligned} &K_+(\mathcal{G})^{1-\theta}K_\times(\mathcal{G})^\theta\sqrt{N_1N_2} = \\ &|A_1 \underset{\mathcal{G}}{+} A_2|^{1-\theta}|A_1 \underset{\mathcal{G}}{+} A_2|^\theta \geq K_+(\mathcal{G}_{1,1})^{1-\theta}K_\times(\mathcal{G}_{1,1})^\theta\sqrt{M_1M_2}\cdot\beta(m_1m_2,\delta_1,L) \\ &\qquad \geq \beta\big(M_1M_2,\delta_0;K_+(\mathcal{G}_{1,1})\big)\cdot\beta(m_1m_2,\delta_1,L) \end{aligned} \tag{3.42}$$

The last inequality is by (3.32).

Recall that, by (3.39)

$$(M_1M_2)(m_1m_2) \sim \bar{\bar{N}}_1\bar{\bar{N}}_2 > \frac{\delta^5}{(\log\frac{K}{\delta})^3}N \tag{3.43}$$

and by (3.20) and (3.28)

$$M_1 M_2 \lesssim \left(\delta^{-10} K^5 \frac{N_1}{N^{1/4}}\right) \cdot \left(\delta^{-5} K^2 \frac{N_2}{N^{1/4}}\right) \lesssim \delta^{-15} K^7 N^{1/2}. \tag{3.44}$$

By (3.33) and (3.35)

$$\delta_0 \cdot \delta_1 > c\left(\log \frac{K}{\delta}\right)^{-4} \delta \tag{3.45}$$

$$K_+(\mathcal{G}_{1,1}) \cdot L < \delta^{-3} (\log K)^2 K. \tag{3.46}$$

The only missing property at this point is the upper bound (3.7) on $m_1 m_2$. We will achieve this with one more decomposition.

Let $B_i = A_i(\bar{x}_i)$.

For fixed $(\bar{x}_1, \bar{x}_2) \in \mathcal{G}_{1,1}$, consider the graph $\mathcal{K} = \mathcal{G}_{\bar{x}_1, \bar{x}_2} \subset A_1(\bar{x}_1) \times A_2(\bar{x}_2) \subset \mathcal{R}_2 \times \mathcal{R}_2$ satisfying by (3.34) and (3.36)

$$\mathcal{K} \subset B_1 \times B_2 \subset \mathcal{R}_2 \times \mathcal{R}_2$$

$$|B_i| \sim m_i, \quad i = 1, 2,$$

$$|\mathcal{K}| \sim \delta_1 m_1 m_2$$

$$K_+(\mathcal{K}) \sim L.$$

Repeat the process in Claims 1-4 to the graph $\mathcal{K}$ with respect to the decomposition $\mathcal{R}_2 = \mathbb{R} \times \prod_{t'+2}^{t} \mathbb{R}$ with $\pi_2 \colon \mathcal{R}_2 \to \mathbb{R}$ being the projection to the first coordinate. Thus $\mathcal{K}$ gets replaced by (cf. (3.36)-(3.39))

$$\tilde{\mathcal{K}} = \bigcup_{(z_1, z_2) \in \mathcal{K}_{1,1}} (z_1, z_2) \times \mathcal{K}_{z_1, z_2}$$

where

$$\mathcal{K}_{1,1} \subset \mathbb{R} \times \mathbb{R},$$

$$\mathcal{K}_{z_1, z_2} \subset \bar{\bar{B}}_1(z_1) \times \bar{\bar{B}}_2(z_2).$$

Also, (3.18), (3.19) and (3.6) give

$$m_i = |B_i| \geq |\bar{\bar{B}}_i| := \bar{\bar{m}}_i > \frac{\delta_1^3}{(\log \frac{L}{\delta_1})^2} m_i \tag{3.47}$$

$$|\bar{\bar{B}}_i(z_i)| \sim \ell_i \leq |B_i(z_i)| = |A_i(\bar{x}_i, z_i)| < (N_1 N_2)^{1/4}. \tag{3.48}$$

$$|\mathcal{K}_{z_1,z_2}| \sim \delta_3 \ell_1 \ell_2$$
$$|\mathcal{K}_{1,1}| > \frac{\delta_1}{\delta_3 (\log \frac{L}{\delta_1})^4} \, \frac{\bar{\bar{m}}_1 \bar{\bar{m}}_2}{\ell_1 \ell_2}$$

(cf. (3.32), (3.33))

$$K_+(\mathcal{K}_{z_1,z_2}) < K_+(\mathcal{K}_{1,1}) \cdot K_+(\mathcal{K}_{z_1,z_2}) < \delta_1^{-3} (\log L)^2 L \tag{3.49}$$

$\big($cf. (3.35)$\big)$.

(We point out here that $\ell_i, \bar{\bar{m}}_i, \delta_3 > \frac{\delta_1}{(\log \frac{L}{\delta_1})4}$ do depend on the basepoint $(\bar{x}_1, \bar{x}_2) \in \mathcal{R}_1 \times \mathcal{R}_1$).

To estimate $\beta(m_1 m_2, \delta_1, L)$ in (3.42), we will give a lower bound on

$$K_+(\mathcal{K})^{1-\theta} K_\times(\mathcal{K})^\theta \sqrt{m_1 m_2}.$$

First, we remark that from (3.45) and (3.46), we have

$$\delta_1 > \frac{\delta}{(\log \frac{K}{\delta})^4}, \tag{3.50}$$

$$L < \frac{K (\log K)^2}{\delta^3} < \Big(\frac{K}{\delta}\Big)^3, \tag{3.51}$$

and

$$\frac{L}{\delta_1} < \frac{K (\log K)^2}{\delta^3} \frac{(\log \frac{K}{\delta})^4}{\delta} < \Big(\frac{K}{\delta}\Big)^4. \tag{3.52}$$

On the other hand, applying Lemma 2.1 to $\mathcal{K}_{1,1} \subset \mathbb{R} \times \mathbb{R}$, we have

$$K_+(\mathcal{K}_{1,1})^{1-\theta} K_\times(\mathcal{K}_{1,1})^\theta > \left[\frac{\delta_1}{\delta_3 (\log L/\delta_1)^4}\right]^2 \left(\frac{\bar{\bar{m}}_1 \, \bar{\bar{m}}_2}{\ell_1 \ell_2}\right)^{\theta/4} \tag{3.53}$$

Also, note that, from (3.48)

$$\ell_1 \ell_2 < N^{1/2}. \tag{3.54}$$

Thus

$$\begin{aligned}
&K_+(\mathcal{K})^{1-\theta}K_\times(\mathcal{K})^\theta\sqrt{m_1m_2}\\
&=|B_1 \underset{\mathcal{K}}{+} B_2|^{1-\theta}|B_1 \underset{\mathcal{K}}{\times} B_2|^\theta\\
&\geq |\ \bar{\bar{B}}_1 \underset{\tilde{\mathcal{K}}}{+}\bar{\bar{B}}_2|^{1-\theta}|\ \bar{\bar{B}}_1 \underset{\tilde{\mathcal{K}}}{\times}\bar{\bar{B}}_2|^\theta\\
&\geq K_+(\mathcal{K}_{1,1})^{1-\theta}K_\times(\mathcal{K}_{1,1})^\theta\left(\frac{\bar{\bar{m}}_1\bar{\bar{m}}_2}{\ell_1\cdot\ell_2}\right)^{\frac{1}{2}}\beta\big(\ell_1\ell_2,\delta_3,\delta_1^{-3}(\log L)^2L\big)\\
&>\frac{\delta_1^2}{\delta_3^2(\log\frac{L}{\delta}{}_1)^8}\left(\frac{\bar{\bar{m}}_1\bar{\bar{m}}_2}{\ell_1\ell_2}\right)^{\frac{1}{2}+\frac{\theta}{4}}\beta\big(\ell_1\ell_2,\delta_3,\delta_1^{-3}(\log L)^2L\big)\\
&>\frac{\delta_1^6}{(\log\frac{L}{\delta}{}_1)^{11}}\left(\frac{m_1m_2}{\ell_1\ell_2}\right)^{\frac{1}{2}+\frac{\theta}{4}}\beta\big(\ell_1\ell_2,\delta_3,\delta_1^{-3}(\log L)^2L\big)\\
&>\delta^6(\log\frac{K}{\delta})^{-35}\left(\frac{m_1m_2}{\ell_1\ell_2}\right)^{\frac{1}{2}+\frac{\theta}{4}}\beta\Big(\ell_1\ell_2,\frac{\delta_1}{(\log\frac{K}{\delta})^4},\delta^{-3}\big(\log\frac{K}{\delta}\big)^2L\Big)\\
&>\min_{N''}\Big\{\delta^6(\log\frac{K}{\delta})^{-35}\left(\frac{m_1m_2}{N''}\right)^{\frac{1}{2}+\frac{\theta}{4}}\beta\Big(N'',\frac{\delta_1}{(\log\frac{K}{\delta})^4}\delta^{-3}\big(\log\frac{K}{\delta}\big)^2L\Big)\Big\},
\end{aligned}\tag{3.55}$$

where the minimum is taken over all $N'' < \min\{m_1m_2, N^{\frac{1}{2}}\}$. starting from the second inequality, we use (3.49), (3.53), (3.47), (3.50)-(3.52), (3.54).

We replace in (3.42), $\beta(m_1m_2,\delta_1,L)$ by (3.55) and set

$$N'=M_1M_2,\ \delta'=\delta_0,\ \delta''=\frac{\delta_1}{(\log\frac{K}{\delta})^4},\ K'=K_+(\mathcal{G}_{1,1}),\ K''=(\log\frac{K}{\delta})^2\delta^{-3}L$$

Using (3.43), we get the following estimate.

$$\begin{aligned}
\beta(N,\delta,K)&>\delta^6(\log\frac{K}{\delta})^{-35}\ \beta(N',\delta',K')\cdot\beta(N'',\delta'',K'')\Big(\frac{\delta^5}{(\log\frac{K}{\delta})^3}\frac{N}{N'N''}\Big)^{\frac{1}{2}+\frac{\theta}{4}}\\
&>\frac{\delta^{11}}{(\log\frac{K}{\delta})^{38}}\ \beta(N',\delta',K')\cdot\beta(N'',\delta'',K'')\cdot\left(\frac{N}{N'N''}\right)^{\frac{1}{2}+\frac{\theta}{4}},
\end{aligned}$$

where, by (3.44), (3.55), (3.45) and (3.46),

$$N', N'' < \left(\frac{K}{\delta}\right)^{15} N^{1/2}$$

$$\delta' \cdot \delta'' > \left(\log \frac{K}{\delta}\right)^{-8} \delta$$

$$K' \cdot K'' < \delta^{-6} \left(\log \frac{K}{\delta}\right)^{16} K. \qquad \square$$

This proves Lemma 3.1.

Ignoring the dependence on K, define

$$\beta(N, \delta) = \beta_\theta(N, \delta) = \min\{K_+(\mathcal{G})^{1-\theta} K_\times(\mathcal{G})^\theta N^{\frac{1}{2}}\},$$

where the minimum is taken over all $A_1, A_2 \subset \mathcal{R}, \mathcal{G} \subset A_1 \times A_2$ such that

$$|A_i| = N_i, N = N_1 N_2, |\mathcal{G}| > \delta N.$$

Thus $\beta(N, \delta) = \min_K \beta(N, \delta, K)$.

Corollary 3.1.1. *Let $0 < \theta < 10^{-3}$ be a constant. Then*

$$\beta(N, \delta) > \min \left\{ \delta N^{\frac{1}{2}+\frac{1}{120}}, \delta^{11} (\log N)^{-38} \beta(N', \delta') \beta(N'', \delta'') \left(\frac{N}{N'N''}\right)^{\frac{1}{2}+\frac{\theta}{4}} \right\}$$

where the minimum is taken over

$$N', N'' < N^{5/8}, N'N'' < N$$

$$\delta' \cdot \delta'' > (\log N)^{-8} \delta.$$

Proof. We distinguish 2 cases.

If $\frac{K_+(\mathcal{G})}{\delta} > N^{\frac{1}{120}}$, obviously $K_+(\mathcal{G})^{1-\theta} K_\times(\mathcal{G})^\theta N^{\frac{1}{2}} > \delta N^{\frac{1-\theta}{120}} N^{-\frac{\theta}{2}} N^{\frac{1}{2}} > \delta N^{\frac{1}{2}+\frac{1}{120}}$ by assumption on θ.

If $\frac{K_+(\mathcal{G})}{\delta} < N^{\frac{1}{120}}$, apply (3.5) with $K = K_+(\mathcal{G})$. We obtain the lower bound

$$\delta^{11} (\log N)^{-38} \beta(N', \delta') \beta(N'', \delta'') \left(\frac{N}{N', N''}\right)^{\frac{1}{2}+\frac{\theta}{4}}$$

with $N', N'', \delta', \delta''$ subject to the constrains

$$N'N'' < N; N', N'' < N^{\frac{1}{2}+\frac{1}{8}}$$

$$\delta' \cdot \delta'' > (\log N)^{-4}\delta$$

from (3.5), (3.6). □

For technical reason, we redefine $\beta_\theta(N, \delta, K)$ and $\beta_\theta(N, \delta)$ by taking

$$\tilde{\beta}_\theta(N, \delta, K) = \min_{M<N} \left(\frac{N}{M}\right)^{\frac{1}{2}+\frac{\theta}{4}} \beta_\theta(M, \delta, K) \tag{3.56}$$

and

$$\tilde{\beta}_\theta(N, \delta) = \min_{M<N} \left(\frac{N}{M}\right)^{\frac{1}{2}+\frac{\theta}{4}} \beta_\theta(M, \delta). \tag{3.57}$$

Lemma 3.1 and Corollary 3.1.1 may then be restated in the following simpler form.

Lemma 3.2. *Let $0 < \theta < 10^{-3}$ be a constant.*

$$\tilde{\beta}(N, \delta, K) > \min \frac{\delta^{11}}{(\log \frac{K}{\delta})^{38}} \tilde{\beta}(N', \delta', K') \cdot \tilde{\beta}(N'', \delta'', K'')$$

with minimum taken over

$$\left(\frac{K}{\delta}\right)^{-15} N^{1/2} < N', N'' < \left(\frac{K}{\delta}\right)^{15} N^{1/2}; N \sim N'N'' \tag{3.58}$$

$$\delta' \cdot \delta'' > \left(\log \frac{K}{\delta}\right)^{-8} \delta \tag{3.59}$$

$$K' \cdot K'' < \delta^{-6} \left(\log \frac{K}{\delta}\right)^{16} K. \tag{3.60}$$

Lemma 3.3. *Let $0 < \theta < 10^{-3}$ be a constant.*

$$\tilde{\beta}(N, \delta) > \min \delta^{11} (\log N)^{-38} \cdot \tilde{\beta}(N', \delta') \tilde{\beta}(N'', \delta'')$$

with minimum taken over

$$N', N'' < N^{5/8}, \; N \sim N'N''$$

$$\delta' \cdot \delta'' > (\log N)^{-4} \delta.$$

4. Finite Products

Assume $\mathcal{G} \subset A_1 \times A_2$ where $A_i \subset \prod_1^t \mathbb{R}$.

Denote

$$\tilde{\beta}^{(t)}(N, \delta)$$

the quantity (3.39), but under the restriction of an index set of size t. Going back to the proof of the factorization Lemma 3.1, we split the index set into $\{1, \cdots, t'\} \cup \{t' + 1\} \cup \{t' + 2, \cdots, t\}$. Hence Lemma 3.3 may be restated as

Lemma 4.1.

$$\tilde{\beta}^{(t)}(N, \delta) > \min \delta^{11} (\log N)^{-38} \tilde{\beta}^{(t')}(N', \delta') \tilde{\beta}^{(t'')}(N'', \delta'') \tag{4.1}$$

with minimum taken over

$$t' + t'' t', t'' < t \tag{4.2}$$

$$N', N'' < N^{5/8}, N = N'N''$$

$$\delta' \cdot \delta'' > (\log N)^{-8} \delta. \tag{4.3}$$

Lemma 4.2. *Let* $0 < \theta < 10^{-3}$ *be a constant. Then*

$$\tilde{\beta}^{(t)}(N, \delta) > \delta^{11t} (\log N)^{-45t^2} N^{\frac{1}{2} + \frac{\theta}{4}}.$$

Proof.

We proceed by induction on t.

If $t = 1$. Lemma 2.1 gives $\beta^{(1)}(N, \delta) > \delta^2 N^{\frac{1}{2} + \frac{\theta}{4}}$.

By (4.2), (4.3)

$$(\delta')^{11t'} (\delta'')^{11t''} \geq (\delta' \delta'')^{11(t-1)} > (\log N)^{-88(t-1)} \delta^{11(t-1)}$$

For Lemma 4.1 and inductive assumption for $t', t'' < t$, it follows that right hand side of (4.1) is at least

$$\begin{aligned} &\delta^{11} (\log N)^{-38} (\delta')^{11t'} (\log N')^{-45(t')^2} (\delta'')^{11t''} (\log N'')^{-45(t'')^2} N^{\frac{1}{2} + \frac{\theta}{4}} \\ &> \delta^{11t} (\log N)^{-38 - 45(1 + (t-1)^2) - 88(t-1)} N^{\frac{1}{2} + \frac{\theta}{4}} \\ &> \delta^{11t} (\log N)^{-45t^2} N^{\frac{1}{2} + \frac{\theta}{4}}. \qquad \square \end{aligned}$$

5. Use of Freiman's Lemma

Dimensional reduction in terms of additive doubling constant will be achieved using Freiman's Lemma.

Lemma 5.1. *(Freiman): If A is a finite subset of a real vector space E satisfying $|A+A| \leq K|A|$, then* $\dim[A] \leq K$.

It follows that if $A \subset \mathcal{R} = \prod \mathbb{R}$ satisfies $|A| < \infty, |A+A| \leq K|A|$, then after reorganizing the index set, the restriction of the coordinate map $\pi|_A : \prod \mathbb{R} \to \prod_1^t \mathbb{R}$ is one-to-one on A.

As the first dimensionless lower bound on $\tilde{\beta}(N, \delta, K)$, we obtain

Lemma 5.2. *Let $0 < \theta < 10^{-3}$ be a constant. Then*

$$\tilde{\beta}(N, \delta, K) > (\log N)^{-10^3 (\frac{K}{\delta})^{120}} N^{\frac{1}{2} + \frac{\theta}{4}}.$$

Proof.

Let $\mathcal{G} \subset A_1 \times A_2 \subset \mathcal{R}, |\mathcal{G}| > \delta N_1 N_2$.

Assume $N_1 \geq N_2$. By (2.5), since $K_+(\mathcal{G}) \leq K$

$$N_2 > \left(\frac{\delta}{K}\right)^2 N_1.$$

Let $A = A_1 \cup A_2$ and consider $\mathcal{G} \subset A \times A$. Thus $|A| \sim N_1$ and

$$|\mathcal{G}| > \frac{\delta^3}{K^2} N_1^2 := \delta_1 N_1^2 \tag{5.1}$$

$$|A \underset{\mathcal{G}}{+} A| \leq K N_1^2. \tag{5.2}$$

From (5.1), (5.2) and the Balog-Szemerédi-Gowers theorem, there is a subset $A' \subset A$ satisfying the properties

$$|A' + A'| < \left(\frac{K}{\delta_1}\right)^{20} |A'| < \left(\frac{K}{\delta}\right)^{60} |A'| \tag{5.3}$$

$$|(A' \times A') \cap \mathcal{G}| > \left(\frac{\delta_1}{K}\right)^{20} N_1^2 > \left(\frac{\delta}{K}\right)^{60} N_1^2. \tag{5.4}$$

Hence

$$|A'| > \left(\frac{\delta}{K}\right)^{60} N_1. \tag{5.5}$$

From (5.3) and Lemma 5.1, there is an index set of size t

$$t < \left(\frac{K}{\delta}\right)^{60} \tag{5.6}$$

and $\pi|_{A'}$ is one-to-one. Denoting $\mathcal{G}' = (A' \times A') \cap \mathcal{G}$ and $\mathcal{H} = (\pi \times \pi)(\mathcal{G}') \subset \pi(A') \times \pi(A')$, by (5.4), (5.6), (4.7) and (5.5), we get

$$\begin{aligned}
|A_1 \underset{\mathcal{G}}{+} A_2|^{1-\theta}|A_1 \underset{\mathcal{G}}{\times} A_2|^{\theta} &\geq |A' \underset{\mathcal{G}'}{+} A'|^{1-\theta}|A' \underset{\mathcal{G}'}{\times} A'|^{\theta} \\
&\geq |\pi(A') \underset{\mathcal{H}}{+} \pi(A')|^{1-\theta}|\pi(A') \underset{\mathcal{H}}{\times} \pi(A')|^{\theta} \\
&\geq \tilde{\beta}^{(t)}\left(|A'|^2, \left(\frac{\delta}{K}\right)^{60}\right) \\
&> \left(\frac{\delta}{K}\right)^{660t} (\log N)^{-45t^2} |A'|^{1+\frac{\theta}{2}} \\
&> \left(\frac{\delta}{K}\right)^{10^3 t} (\log N)^{-45t^2} N_1^{1+\frac{\theta}{2}}.
\end{aligned}$$

Therefore, (5.6) implies

$$\beta(N_1 N_2, \delta, K) \geq \left(\frac{\delta}{K}\right)^{10^3(\frac{K}{\delta})^{60}} (\log N)^{-45(\frac{K}{\delta})^{120}} (N_1 N_2)^{\frac{1}{2}+\frac{\theta}{4}}$$

and also

$$\tilde{\beta}(N, \delta, K) > (\log N)^{-10^3(\frac{K}{\delta})^{120}} N^{\frac{1}{2}+\frac{\theta}{4}}.$$

This proves (5.2).

Dependence of (5.2)-estimate on K is very poor. Next we get an improved behavior combining (5.2) and (3.45).

6. First Improvement

We establish the following improvement of Lemma 5.2.

Lemma 6.1. *Let $0 < \theta < 10^{-3}$ be a constant. Then*

$$\tilde{\beta}(N, \delta, K) > (\log N)^{-(\log \frac{K}{\delta})^{C/\theta}} N^{\frac{1}{2}+\frac{\theta}{8}}. \tag{6.1}$$

Thus the dependence on K/δ is considerably improved.

Proof. We will make an iterated application of Lemma 3.1.

Fix N, δ, K and choose an integer t of the form 2^ℓ (to be specified). Starting from the expression

$$\phi(N,\delta,K) = \phi_o(N,\delta,K) = (\log N)^{-10^3(\frac{K}{\delta})^{120}} N^{\frac{1}{2}+\frac{\theta}{4}} + 1 \tag{6.2}$$

obtained in Lemma 5.2, define recursively for $\ell' = 0, 1, \dots, \ell - 1$

$$\phi_{\ell'+1}(N,\delta,K) = \delta^{11}(\log \frac{K}{\delta})^{-38} \min \phi_{\ell'}(N',\delta',K')\phi_{\ell'}(N'',\delta'',K'') \tag{6.3}$$

with $N', N'', \delta', \delta'', K', K''$ subject to restrictions (3.67)-(3.69).

We evaluate $\tilde{\phi} = \phi_\ell$.

Iterating (6.3), we obtain clearly

$$\tilde{\phi}(N,\delta,K) = \prod_{\nu \in \underset{\ell'<\ell}{\cup} \{0,1\}^{\ell'}} \delta_\nu^{11}(\log \frac{K_\nu}{\delta_\nu})^{-38} \prod_{\nu\in\{0,1\}^\ell} \phi(N_\nu,\delta_\nu,K_\nu) \tag{6.4}$$

where $(N_\nu)_{\nu\in \underset{\ell'\leq\ell}{\cup}\{0,1\}^{\ell'}}, (\delta_\nu)_{\nu\in \underset{\ell'\leq\ell}{\cup}\{0,1\}^{\ell'}}$ satisfy by (3.67)-(3.48) the following constraints

$$N_\phi = N, \delta_\phi = \delta, K_\phi = K$$

$$N_\nu \sim N_{\nu,0}\cdot N_{\nu,1} \tag{6.5}$$

$$N_{\nu,0} + N_{\nu,1} \leq \left(\frac{K_\nu}{\delta_\nu}\right)^{15} N_\nu^{1/2} \tag{6.6}$$

$$\delta_{\nu,0}\cdot\delta_{\nu,1} \geq \left(\log\frac{K_\nu}{\delta_\nu}\right)^{-4}\delta_\nu \tag{6.7}$$

$$K_{\nu,0}, K_{\nu,1} < \delta_\nu^{-6}\left(\log\frac{K_0}{\delta_0}\right)^4 K_\nu. \tag{6.8}$$

From (6.7), (6.8)

$$\log\frac{K_{\nu,0}}{\delta_{\nu,0}} + \log\frac{K_{\nu,1}}{\delta_{\nu,1}} < 8\log\frac{K_\nu}{\delta_\nu}$$

and iteration implies

$$\max_{\nu\in\{0,1\}^{\ell'}} \log\frac{K_\nu}{\delta_\nu} \leq \sum_{\nu\in\{0,1\}^{\ell'}} \log\frac{K_\nu}{\delta_\nu} < 8^{\ell'}\log\frac{K}{\delta}. \tag{6.9}$$

Iteration of (6.7) gives

$$\prod_{\nu\in\{0,1\}^{\ell'}} \delta_\nu > \prod_{\nu\in\{0,1\}^{\ell'-1}} \left(\log \frac{K_\nu}{\delta_\nu}\right)^{-4} \prod_{\nu\in\{0,1\}^{\ell'-1}} \delta_\nu$$

$$> 8^{-2\ell' 2^{\ell'}} \left(\log \frac{K}{\delta}\right)^{-2\cdot 2^{\ell'}} \prod_{\nu\in\{0,1\}^{\ell'-1}} \delta_\nu$$

$$> 8^{-2(\ell' 2^{\ell'}+(\ell'-1)2^{\ell'-1}+\cdots)} \left(\log \frac{K}{\delta}\right)^{-2(2^{\ell'}+2^{\ell'-1}+\cdots)} \delta$$

$$> 8^{-4\ell' 2^{\ell'}} \left(\log \frac{K}{\delta}\right)^{-4\cdot 2^{\ell'}} \delta. \tag{6.10}$$

The second inequality follows from (6.9).

Next, iterate (6.8). Thus, by (6.9) and (6.10) that

$$\prod_{\nu\in\{0,1\}^{\ell'}} K_\nu \leq \prod_{\nu\in\{0,1\}^{\ell'-1}} \delta_\nu^{-6} (\log K_\nu)^4 \prod_{\nu\in\{0,1\}^{\ell'-1}} K_\nu$$

$$< \left(8^{-2\ell' 2^{\ell'}} \left(\log \frac{K}{\delta}\right)^{-2\cdot 2^{\ell'}} \delta\right)^{-6} \left(8^{\ell'} \log \frac{K}{\delta}\right)^{2\cdot 2^{\ell'}} \left(\prod_{\nu\in\{0,1\}^{\ell'-1}} K_\nu\right)$$

$$< 8^{14\cdot \ell' 2^{\ell'}} \left(\log \frac{K}{\delta}\right)^{14\cdot 2^{\ell'}} \delta^{-6\ell'} K. \tag{6.11}$$

From (6.5)

$$\prod_{\nu\in\{0,1\}^{\ell}} N_\nu > C^{-2^{\ell}} N. \tag{6.12}$$

From (6.7) (which implies that $\delta_{\nu,0}, \delta_{\nu,1} > (\log \frac{K_\nu}{\delta_\nu})^{-4}\delta_\nu$) and (6.9) that

$$\delta_\nu > 8^{-4\ell^2} \left(\log \frac{K}{\delta}\right)^{-4\ell} \delta \tag{6.13}$$

and from (6.8) (which implies that $K_{\nu,0}, K_{\nu,1} \leq \delta_\nu^{-6}(\log K_\nu)^4 K_\nu$), (6.9) and (6.13) that

$$K_\nu < 8^{25\ell^3} \left(\log \frac{K}{\delta}\right)^{25\ell^2} \delta^{-6\ell} K. \tag{6.14}$$

From (6.6), (6.13), (6.14)

$$N_{\nu,0} + N_{\nu,1} \leq 8^{450\ell^3} \left(\log \frac{K}{\delta} \right)^{450\ell^2} \delta^{-90\ell} K^{15} N_\nu^{1/2}$$

hence

$$N_\nu < 10^{10^3\ell^3} \left(\log \frac{K}{\delta} \right)^{10^3\ell^2} \delta^{-10^3\ell} K^{30} N^{1/t}. \tag{6.15}$$

From (6.2), (6.4), (6.9), (6.10)

$$\tilde{\phi}(N, \delta, K) \geq 8^{-44\ell 2^\ell} \left(\log \frac{K}{\delta} \right)^{-44\, 2^\ell} \delta^{11\ell} \left(8^\ell \log \frac{K}{\delta} \right)^{-38\, 2^\ell} \prod_{\nu \in \{0,1\}^\ell} \phi(N_\nu, \delta_\nu, K_\nu)$$

$$> \left(8^\ell \log \frac{K}{\delta} \right)^{-82\, 2^\ell} \delta^{11\ell} \prod_{\nu \in \{0,1\}^\ell} [1 + (\log N)^{-10^3 (\frac{K_\nu}{\delta_\nu})^{120}} N_\nu^{\frac{1}{2} + \frac{\theta}{4}}] \tag{6.16}$$

To control the last factor in the expression above, we decompose

$$\{0,1\}^\ell = I \cup J$$

with

$$I = \{\nu \in \{0,1\}^\ell \Big| \frac{K_\nu}{\delta_\nu} < A\}$$

and A to be specified.

First, we want to bound $|J|$.

By (6.10), (6.11)

$$A^{|J|} < \prod_{\nu \in \{0,1\}^\ell} \frac{K_\nu}{\delta_\nu} < \left(8^\ell \log \frac{K}{\delta} \right)^{18t} \delta^{-7\ell} K. \tag{6.17}$$

Take

$$2^\ell = t \sim \log \frac{K}{\delta} \tag{6.18}$$

and fixing $0 < \gamma < 1$, take

$$\log A \sim \gamma^{-1} t. \tag{6.19}$$

With this choice, (6.17) implies

$$|J| < \frac{10^3 t \log t}{\log A} < \gamma t.$$

Thus

$$\prod_{\nu\in\{0,1\}^\ell} 1+(\log N)^{-10^3(\frac{K_\nu}{\delta_\nu})^{120}} N_\nu^{\frac12+\frac\theta4}$$

$$> (\log N)^{-10^3A^{120}2^\ell}\Big(\prod_{\nu\in I} N_\nu\Big)^{\frac12+\frac\theta4}$$

$$> c'(\log N)^{-10^3A^{120}t}[10^{10^3\ell^3}(\log\frac{K}{\delta})^{10^3\ell^2}\delta^{-10^3\ell}K^{30}N^{1/t}]^{-|J|}N^{\frac12+\frac\theta4}$$

$$> (\log N)^{-10^3A^{120}t}10^{-10^3\gamma t(\log t)^3}(\log N)^{-10^3\gamma t(\log t)^2}\delta^{10^3\gamma t\log t}N^{\frac12+\frac\theta4-\gamma}.$$

The second inequality follows from (6.12) and (6.15).

Thus by (6.16) and (6.18), (6.19), letting $\gamma=\frac\theta8$

$$\tilde\phi(N,\delta,K) > (\log N)^{-t^{C/\gamma}}\cdot N^{\frac12+\frac\theta4-\gamma}$$

$$> (\log N)^{-(\log\frac K\delta)^{C/\theta}}\ N^{\frac12+\frac\theta8}$$

which is (6.1).

Remark. Notice that proof of (6.1) relies on Lemma 3.2, Replacing (3.47) by the cruder bound $\delta'\delta''>\frac{\delta}{(\log N)^4}$, we would obtain the bound $(\log N)^{-(\log N)^{C/\theta}}N^{\frac12+\frac\theta8}$ in (6.1), which is useless.

7. Sum-Product Theorem in $\mathcal R$ b We prove the following

Lemma 7.1. *Fix a constant $0<\theta<10^{-3}$. There are positive constants b_1,b_2,b_3 such that*

$$\tilde\beta(N,\delta,K) > K^{-b_1}\delta^{b_2\log\log N}\ e^{b_3(\log\log N)^2}\ N^{\frac12+\frac{\theta}{10}}. \tag{7.1}$$

Proof.

We proceed in 2 steps.

Choose a large integer $\tilde N$ and let

$$(\log\bar N)^{1-\frac{\theta}{3C}}=b_1<b_2<b_3\sim(\log\bar N)^{1-\frac{\theta}{3C}} \tag{7.2}$$

where C is the constant in (6.1). The precise choice of b_1,b_2,b_3 will be specified later. We verify (7.1) assuming $\log N\sim\log\bar N$.

We distinguish 2 cases.

(i) $\log \frac{K}{\delta} < (\log \bar{N})^{\frac{\theta}{2C}}$

For $\bar{N}$ large enough, (6.1) gives

$$\begin{aligned}\tilde{\beta}(N, \delta, K) &> (\log N)^{-(\log \frac{K}{\delta})^{C/\theta}} \; N^{\frac{1}{2}+\frac{\theta}{8}}. \qquad (7.4)\\ &\gtrsim (\log \bar{N})^{-(\log \bar{N})^{1/2}} \; N^{\frac{1}{2}+\frac{\theta}{8}}\\ &> e^{b_3(\log\log N)^2} \; N^{\frac{1}{2}+\frac{\theta}{10}},\end{aligned}$$

which is bigger than the right hand side of (7.1). The last inequality is by (7.2)

(ii) $\log \frac{K}{\delta} \geq (\log \bar{N})^{\frac{\theta}{2C}}$

Again, by (7.2), the right hand side of (7.1) is

$$\begin{aligned}(7.1) &< \left(\frac{\delta}{K}\right)^{(\log \bar{N})^{1-\frac{\theta}{3C}}} \qquad e^{b_3(\log\log N)^2} \; N^{\frac{1}{2}+\frac{\theta}{10}}\\ &< e^{-(\log \bar{N})^{1+\frac{\theta}{6C}}} \bar{N} < 1\end{aligned}$$

so that inequality (7.1) becomes trivial.

Next, having (7.1) for $\log N \sim \log \bar{N}$, we verify (7.1) for all $N \geq \bar{N}$ using Lemma 3.2 and induction on the size of N.

Thus, according to Lemma 3.2

$$\tilde{\beta}(N, \delta, K) > \delta^{11} (\log N)^{-38} \; \tilde{\beta}(N', \delta', K') \cdot \tilde{\beta}(N'', \delta'', K'') \tag{7.3}$$

where

$$N \sim N'N'', \left(\frac{K}{\delta}\right)^{-15} N^{1/2} < N', N'' < \left(\frac{K}{\delta}\right)^{15} N^{1/2} \tag{7.4}$$

$$\delta'\delta'' > (\log N)^{-4}\delta \qquad (7.5))$$

$$K'K'' < \delta^{-6}(\log N)^4 K. \tag{7.6}$$

We may obviously assume $\frac{K}{\delta} < N^{10^{-4}}$ since otherwise (7.1) is trivial. From (7.4), we get then $N', N'' < N^{3/5}$ for which the validity of (7.1) is assumed (notice that if $N \geq \bar{N}$, $\log N' \sim \log N'' \gtrsim \log \bar{N}$).

Since $N^{2/5} < N', N'' < N^{3/5}$,(using '$\ell\ell$' to denote $\log\log$)

$$\ell\ell N - \log\frac{5}{2} < \ell\ell N', \ell\ell N'' < \ell\ell N - \log\frac{5}{3}.$$

Thus

$$\begin{aligned}(\delta')^{b_2\ell\ell N'}(\delta'')^{b_2\ell\ell N''} &> (\delta'\delta'')^{b_2\ell\ell N - b_2\log\frac{5}{3}} \\ &> (\log N)^{-8b_2\ell\ell N}\delta^{b_2\ell\ell N - b_2\log\frac{5}{3}}.\end{aligned}$$

The last inequality is by (7.5)

Therefore, (7.3) gives

$$\begin{aligned}&\tilde{\beta}(N,\delta,K)\\ &> \delta^{11}(\log N)^{-38}(K'K'')^{-b_1}(\delta')^{b_2\ell\ell N'}(\delta'')^{b_2\ell\ell N''}e^{b_3[(\ell\ell N')^2+(\ell\ell N'')^2]}(N'N'')^{\frac{1}{2}+\frac{\theta}{10}}\\ &> \delta^{11+6b_1}(\log N)^{-38-4b_1}K^{-b_1}(\delta')^{b_2\ell\ell N'}(\delta'')^{b_2\ell\ell N''}e^{b_3[(\ell\ell N')^2+(\ell\ell N'')^2]}N^{\frac{1}{2}+\frac{\theta}{10}}\\ &> \delta^{11+6b_1-b_2\log\frac{5}{3}}(\log N)^{-38-4b_1-8b_2\ell\ell N}e^{\frac{19}{10}b_3(\ell\ell N)^2}K^{-b_1}\cdot\delta^{b_2\ell\ell N}\cdot N^{\frac{1}{2}+\frac{\theta}{10}}\\ &> K^{-b_1}\delta^{b_2\ell\ell N}e^{b_3(\ell\ell N)^2}N^{\frac{1}{2}+\frac{\theta}{10}}.\end{aligned}$$

The second inequality is by (7.6).

Lemma 7.1 is proved by choosing

$$b_2 = \frac{11+6b_1}{\log\frac{5}{3}}. \qquad \square$$

Theorem 2. *There is an absolute constant $\tau > 0$ such that if $A \subset \mathcal{R} = \prod\mathbb{R}$ is a finite set, with $|A| = M$ large enough, then either $|A+A| > M^{1+\tau}$ or $|A\cdot A| > M^{1+\tau}$.*

Proof. In (7.1), set $\delta = 1, K = \frac{|A+A|}{|A|}, N = M^2$, we have

$$\beta(M^2,1,K) > K^{-b_1}M^{1+\frac{\theta}{5}}.$$

Hence,

$$\begin{aligned}|A+A|^{1-\theta}|A\cdot A|^{\theta} = K_+(A\times A)^{1-\theta}K_\times(A\times A)^{\theta}M &\geq \beta(M^2,1,K)\\ &> K^{-b_1}M^{1+\frac{\theta}{5}}\\ &= \left(\frac{M}{|A+A|}\right)^{b_1}M^{1+\frac{\theta}{5}}\end{aligned}$$

Therefore

$$|A+A|^{1-\theta+b_1}|A\cdot A|^{\theta} > M^{1+b_1+\frac{\theta}{5}},$$

and

$$\max(|A+A|,|A\cdot A|) > M^{1+\frac{\theta}{5(1+b_1)}}.$$

The theorem is proved by taking $\tau = \frac{\theta}{5(1+b_1)}$. $\square$

Remark. In the proof of Theorem 2, the only place we use the assumption $A \subset \mathbb{R}$ is in Proposition 1.1. If we accept Toth's proof of the Szemerédi-Trotter theorem for the complex plane, statement and proof of Proposition 1.1 are identical. Alternatively, we may adjust the argument from [Ch3] (in the spirit of the original Erdös-Szemerédi proof in [E-S]) to get in the $\mathbb{C}$ case a statement of the form

$$|S \underset{\mathcal{G}}{+} S|\cdot|S \underset{\mathcal{G}}{\times} S| > \delta^{c_1}N^{2+Cc_2} \tag{7.13}$$

for certain constants $c_1, c_2 > 0$. This is much weaker but equally suffices for proving Theorem 2.

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