ON PRODUCT SETS IN SL_2 AND SL_3

Abstract. Here we point out the following two theorems as consequences of the theorems in [C].

1. *For all $\varepsilon > 0$, there is $\delta > 0$ such that if $A \subset SL_3(\mathbb{Z})$ is a finite set, then either*

there exist a nilpotent subgroup G and elements $z_1, \cdots, z_\beta \in SL_3(\mathbb{Z})$ with $\beta < |A|^\varepsilon$ such that

$$A \subset \bigcup_{1 \le i \le \beta} z_i G,$$

or $|A^2| > |A|^{1+\delta}$.

2. *Let $A \subset SL_2(\mathbb{C})$ be a finite set, and $|A^2| < K|A|$. Then there exist a virtually abelian subgroup G of $SL_2(\mathbb{C})$ and elements $z_1, \cdots, z_\beta \in SL_2(\mathbb{C})$, with $\beta \le K^c$, such that*

$$A \subset \bigcup_{1 \le i \le \beta} z_i G$$

In this note we study product theorems for matrix spaces.

Let $A^n = \{a_1 \cdots a_n : a_i \in A, i = 1, \dots, n\}$ be the n-fold product set of A. Recall the following theorems from [C].

Theorem A. *For all $\varepsilon > 0$, there is $\delta > 0$ such that if $A \subset SL_3(\mathbb{Z})$ is a finite set, then one of the following alternatives holds.*

(i) *A intersects a coset of a nilpotent subgroup in a set of size at least $|A|^{1-\varepsilon}$.*

(ii) $|A^3| > |A|^{1+\delta}$.

Theorem B. *Let A be a finite subset of $SL_2(\mathbb{C})$. Then one of the following alternatives holds.*

(i) *A is contained in a virtually abelian subgroup*

(ii) $|A^3| > c|A|^{1+\delta}$ *for some absolute constant $\delta > 0$.*

The purpose of this note is to point out the following consequences from Theorems A and B.

Theorem 1. *For all $\varepsilon > 0$, there is $\delta > 0$ such that if $A \subset SL_3(\mathbb{Z})$ is a finite set, then either*

(1) *there exist a nilpotent subgroup G and elements $z_1, \cdots, z_\beta \in SL_3(\mathbb{Z})$ with $\beta < |A|^\varepsilon$ such that*

$$A \subset \bigcup_{1 \le i \le \beta} z_i G,$$

or

(2) $|A^2| > |A|^{1+\delta}$.

Remark. In particular, in Theorem A (ii), one may replace $|A^3|$ by $|A^2|$.

Proof. Assume (2) fails. Thus $|A^2| \le |A|^{1+\delta}$ for all $\delta > 0$ and in particular

$$E(A, A) \ge \frac{|A|^4}{|A^2|} > |A|^{3-\delta}.$$

Hence, there exists a $|A|^{c\delta}$-approximate group H in $SL_3(\mathbb{Z})$ with the following properties. (See [TV].)

(3) $|A| \le |H| < |A|^{1+c\delta}$.

(4) $|H^3| < |A|^{c\delta}\, |H|$.

(5) $|A \cap Hx_0| > |A|^{-c\delta}|H|$ for some $x_0 \in SL_3(\mathbb{Z})$.

Let $H_0 = Ax_0^{-1} \cap H$. Then (3)-(5) imply

$$|H_0^3| < |H_0|^{1+\frac{2c\delta}{1-c\delta}} < |H_0|^{1+c'\delta}.$$

Hence by Therem A, there exist a nilpotent subgroup G of $SL_3(\mathbb{Z})$ and an element $\xi \in SL_3(\mathbb{Z})$ so that

$$|H_0 \cap \xi G| > |H_0|^{1-\varepsilon}, \tag{6}$$

where $\varepsilon = \varepsilon(\delta) \to 0$, as $\delta \to 0$.

Let $H_1 = H_0 \cap \xi G$ and take a maximal set $\{z_i\}_{1 \le i \le \beta}$ in A such that $z_i H_1 \cap z_j H_1 = \emptyset$ for $i \neq j$. Thus by construction

$$A \subset \bigcup_{1 \le i \le \beta} z_i H_1 H_1^{-1}, \tag{7}$$

where $H_1 H_1^{-1} \subset \xi G (\xi G)^{-1} = \xi G \xi^{-1} := G_1 \simeq G$.

To estimate β, we write

$$\beta = \frac{|\bigcup z_i H_1|}{|H_1|} \le \frac{|A\ Ax_0^{-1}|}{|H_0|^{1-\varepsilon}} \le \frac{|A|^{1+\delta}}{|A|^{(1-c\delta)(1-\varepsilon)}} = |A|^{\varepsilon + (1+c-c\varepsilon)\delta}. \qquad \square \tag{8}$$

Theorem 2. *Let $A \subset SL_2(\mathbb{C})$ be a finite set, and*

$$|A^2| < K|A|. \tag{9}$$

Then there exist a virtually abelian subgroup G of $SL_2(\mathbb{C})$ and elements $z_1, \cdots, z_\beta \in SL_2(\mathbb{C})$, with $\beta \leq K^c$, such that

$$A \subset \bigcup_{1 \leq i \leq \beta} z_i G$$

Proof. Proceeding as before, we get an K^c-approximate group H in $SL_2(\mathbb{C})$ such that

(10) $|A| \leq |H| < K^c|A|$.

(11) $|H^3| < K^c\ |H|$.

(12) $|A \cap Hx_0| > K^{-c}\ |H| \quad$ for some $x_0 \in SL_2(\mathbb{C})$.

Thus, $H_0 = Ax_0^{-1} \cap H$ satisfies $|H_0^3| < K^{2c}|H_0|$. We apply Therem B to the set H_0.

In alternative (ii), $|H_0^3| > c|H_0|^{1+\delta}$, hence by (10)-(12)

$$K > |H_0|^{\frac{\delta}{2c}} > (K^{-c}\ |A|)^{\frac{\delta}{2c}} = K^{-\frac{\delta}{2}}\ |A|^{\frac{\delta}{2c}}$$

and

$$K > |A|^{\frac{1}{c}\frac{\delta}{2+\delta}} > |A|^{\frac{\delta}{3c}}.$$

Obviously,

$$A = \bigcup_{x \in A} x \cdot \{e\},$$

where $\beta = |A| < K^{\frac{3c}{\delta}}$.

In alternative (i), $H_0 \subset G$, where G is virtually abelian.

Again, take a maximal set $\{z_i\}_{1 \leq i \leq \beta}$ in A such that $z_i H_0$ are disjoint. Hence

$$A \subset \bigcup_{1 \leq i \leq \beta} z_i H_0 H_0^{-1} \subset \bigcup_{1 \leq i \leq \beta} z_i G,$$

and

$$\beta \leq \frac{|AH_0|}{|H_0|} \leq \frac{|A\ Ax_0^{-1}|}{|H_0|} = \frac{|A^2|}{|H_0|} < \frac{K|A|}{K^{-c}|A|} = K^{1+c}. \qquad \square$$

References

[C]. M-C. Chang, *Product theorems in SL2 and SL3*, J. Math. Jussieu, 7(1), 1-25, (2008).

[TV]. T. Tao, V. Vu, *Additive Combinatorics*, Cambridge University Press, 2006.