ON A PAPER OF ERDÖS AND SZEKERES

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ABSTRACT. Propositions 1.1 – 1.3 stated below contribute to results and certain problems considered in [E-S], on the behavior of products $\prod_1^n (1-z^{a_j}), 1 \leq a_1 \leq \cdots \leq a_n$ integers. In the discussion below, $\{a_1, \ldots, a_n\}$ will be either a proportional subset of $\{1, \ldots, n\}$ or a set of large arithmetic diameter.

1. **Introduction**

The aim of this paper is to revisit some of the questions put forward in the paper [E-S] of Erdos and Szekeres.

Following [E-S], define

$$M(a_1, \ldots, a_n) = \max_{|z|=1} \prod_{i=1}^{n} |1 - z^{a_i}| \tag{1.1}$$

where we assume $a_1 \leq a_2 \leq \cdots \leq a_n$ positive integers (in this paper, we restrict ourselves to distinct integers $a_1 < \cdots < a_n$).

Denote

$$f(n) = \min_{a_1 \leq \cdots \leq a_n} M(a_1, \ldots, a_n) \quad \text{and} \quad f_*(n) = \min_{a_1 < \cdots < a_n} M(a_1, \ldots, a_n). \tag{1.2}$$

It was proven in [E-S] that

$$f(n) \geq \sqrt{2n}. \tag{1.3}$$

This lower bound remains presently still unimproved.

In the other direction, [E-S] establish an upper bound

$$f(n) < \exp(n^{1-c}) \text{ for some } c > 0. \tag{1.4}$$

Subsequent improvements were given by Atkinson [A]

$$f(n) = \exp\{O(n^{\frac{1}{2}} \log n)\} \tag{1.5}$$

and Odlyzko [O]

$$f(n) = \exp\{O(n^{\frac{1}{3}}(\log n)^{4/3})\}. \tag{1.6}$$

Also to be mentioned is a construction due to Kolountzakis ([Kol2], [Kol4]) of a sequence $1 < a_1 < \cdots < a_n < 2n + O(\sqrt{n})$ for which

$$f_*(n) \leq M(a_1, \dots, a_n) < \exp\{O(n^{\frac{1}{2}} \log n)\} \tag{1.7}$$

(Note that Odlyzko's construction does not come with distinct frequencies).

As shown by Atkinson [A], there is a relation between the [E-S] problem and the *cosine-minimum problem*.

Define

$$M_2(n) = \inf\{-\min_\theta \sum_{j=1}^n \cos a_j\theta\} \tag{1.8}$$

with infinum taken over integer sets $a_1 < \cdots < a_n$.

Then

$$\log f_*(n) < O(M_2(n) \log n). \tag{1.9}$$

The problem of determining $M_2(n)$ was put forward by Ankeny and Chowla [C1] motivated by questions on zeta functions.

It is known that $M_2(n) = O(n^{\frac{1}{2}})$ and conjectured by Chowla that in fact $M_2(n) \sim n^{\frac{1}{2}}$ [C2]. The current best lower bound is due to Ruzsa [R]

$$M_2(n) > \exp(c\sqrt{\log n}) \tag{1.10}$$

for some $c > 0$.

As pointed out in [O], polynomials of the form (1.1) are also of interest in connection to Schinzel's problem [S] of bounding the number of irreducible factors of a polynomial on the unit circle in terms of its degree and L^2-norm.

Propositions 1.1 and 1.2 in this paper establish new results for 'dense' sets $S = \{a_1 < \cdots < a_n\}$. The former improves upon (1.7).

Proposition 1.1. *There is a subset* $\{a_1 < \cdots < a_n\} \subset \{1, \dots, N\}, n \asymp \frac{N}{2}$, *such that*

$$M(a_1, \dots, a_n) < \exp(c\sqrt{n}\sqrt{\log n} \log\log n). \tag{1.11}$$

On the other hand, the following holds

Proposition 1.2. *There is a constant* $\tau > 0$ *such that if* $\{a_1 < \ldots < a_n\} \subset \{1, \ldots, N\}$ *and* $n > (1-\tau)N$*, then*

$$M(a_1, \ldots, a_n) > \exp \tau n. \tag{1.12}$$

The latter result generalizes the comment made in [E-S] that

$$\lim_{n\to\infty} [M(1, 2, \ldots, n)]^{1/n} \tag{1.13}$$

exists and is between 1 and 2.

In converse direction, one may prove new lower bounds on $M(a_1, \ldots, a_n)$ assuming that the set $\{a_1 < \cdots < a_n\}$ has a sufficiently large arithmetic diameter.

First, we are recalling the notion of a '*dissociated set*' of integers. We say that $D = \{\nu_1, \ldots, \nu_m\} \subset \mathbb{Z}$ is dissociated provided D does not admit non-trivial $0, 1, -1$ relations. Thus

$$\varepsilon_1\nu_1 + \cdots + \varepsilon_m\nu_m = 0 \;\text{ with }\; \varepsilon_1 = 0, 1, -1 \tag{1.14}$$

implies

$$\varepsilon_1 = \cdots = \varepsilon_m = 0.$$

A more detailed discussion of this notion and its relation to lacunarity appears in §5 of the paper.

Proposition 1.3. *Assume* $\{a_1 < \cdots < a_n\}$ *contains a dissociated set of size* m*. Then*

$$\log M(a_1, \ldots, a_n) \gg \frac{m^{\frac{1}{2}-\varepsilon}}{(\log n)^{1/2}}. \tag{1.15}$$

Hence (1.15) improves upon (1.3) as soon as

$$m \gg (\log n)^{3+\varepsilon}. \tag{1.16}$$

2. **Preliminary estimates**

Let

$$z = e(\theta) = e^{2\pi i\theta}.$$

By taking the real part of $\mathrm{Log}(1-\mathrm{e}^{2\pi\mathrm{i}\theta}) = -\sum_{\mathrm{k}=1}^{\infty}\frac{1}{\mathrm{k}}\mathrm{e}^{2\pi\mathrm{i}\mathrm{k}\theta}$, we have

$$\log|1-z| = -\sum_{k=1}^{\infty}\frac{\cos 2\pi k\theta}{k}.$$

Therefore, we have

Fact 1.

$$\prod_{j=1}^{n}|1-z^{a_j}| = e^{-\sum_{j=1}^{n}\sum_{k=1}^{\infty}\frac{\cos 2\pi k a_j\theta}{k}}.$$

We first establish some preliminary inequalities for later use.

Since the function e^x is convex, we obtain for any probability measure μ on $\mathbb{T}$ that

$$\prod_{j=1}^{n}|1-e(a_j\theta)| * \mu \geq e^{-(\sum_{j=1}^{n}\sum_{k=1}^{\infty}\frac{\cos 2\pi k a_j\,\cdot}{k})*\mu(\theta)}$$

and therefore we have

Fact 2.

$$\Big\|\prod_{j=1}^{n}|1-e(a_j\theta)|\Big\|_{\infty} \geq e^{-\min_{\theta}\{\sum_{j=1}^{n}\sum_{k=1}^{\infty}\frac{\cos 2\pi k a_j\,\cdot}{k}*\mu\}(\theta)}.$$

Lemma 2.1.

$$\log|1-e^{2\pi i\theta}| \leq -\sum_{j=1}^{J}\frac{\rho^j}{j}\cos 2\pi j\theta + O\Big(\frac{1}{\sqrt{J}}\Big) \tag{2.1}$$

where $\rho = 1-\frac{1}{\sqrt{J}}$ *and (2.1) is valid for all* θ.

Proof. We rely on a calculation that appears in [O], Proposition 1.

Use the inequality $\big($[O], (2.4)$\big)$

$$\Big|\frac{1-e^{i\theta}}{1-\rho e^{i\theta}}\Big| \leq \frac{2}{1+\rho} \text{ for } \theta\in[0,2\pi], 0<\rho<1. \tag{2.2}$$

From (2.2)

$$\begin{aligned}\log|1-e^{i\theta}| &\le \log|1-\rho e^{i\theta}| + \log\frac{2}{1+\rho}\\ &= -\sum_{j=1}^{\infty}\frac{\rho^j}{j}\cos j\theta + \log\frac{2}{1+\rho} \qquad (2.3)\\ &\le -\sum_{j=1}^{J}\frac{\rho^j}{j}\cos j\theta + \frac{\rho^J}{J(1-\rho)} + C(1-\rho)\end{aligned}$$

by partial summation and since

$$\log\frac{2}{1+\rho} = -\log\Big(1-\frac{1-\rho}{2}\Big).$$

Thus (2.1) follows from (2.3) with ρ as above.

□

Proposition 2.2. *There is a subset* $\{a_1 \dots a_m\} \subset \{1, \dots, n\}$ *of size*

$$m \asymp \frac{n}{2}$$

and

$$\Big\|\prod_{k=1}^{m}|1-z^{a_k}|\,\Big\|_{L^\infty(|z|=1)} \le e^{c\sqrt{n}\,\sqrt{\log n})(\log\log n)}. \qquad (2.4)$$

Remark. (2.4) is a slight improvement of the estimate

$$\Big\|\prod_{k=1}^{m}|1-z^{a_k}|\Big\|_{L^\infty(|z|=1)} \le e^{c\sqrt{n}\log n}$$

resulting from a construction in [Kol1], p. 162 of a set $\{a_1, \dots, a_m\}$ as above and such that

$$\sum_{k=1}^{m}\cos 2\pi a_k\theta \ge -c\sqrt{m}$$

and Lemma 2.1

$$\begin{aligned}\log\prod_{k=1}^{m}|(1-2a_k)| &\le -\sum_{j=1}^{J}\frac{\rho^j}{j}\sum_{k=1}^{m}\cos 2\pi a_k(j\theta) + O\Big(\frac{m}{\sqrt{J}}\Big)\\ &\le C(logJ)\sqrt{m} + O\Big(\frac{m}{\sqrt{J}}\Big)\\ &< C\log n\,\sqrt{n},\end{aligned}$$

taking $J = m^2$.

Proof of Proposition 2.2. Take independent selectors $(\xi_j)_{1\leq j<n}$ with values $0, 1$ and mean $\mathbb{E}[\xi_j] = 1 - \frac{j}{n}$. Let $F_n(\theta) = 2\sum_{0<j<n}(1-\frac{j}{n})\cos 2\pi j\theta + 1$ be the Fejer kernel

$$\sum_{k=1}^m \cos a_k\theta = \sum_{\ell=1}^n \xi_\ell \cos \ell\theta = \frac{1}{2}F_n(\theta) - \frac{1}{2} + \sum_{\ell=1}^n (\xi_\ell - \mathbb{E}[\xi_\ell])\cos \ell\theta. \tag{2.5}$$

By Lemma 2.1 (applies with $J = n^{10}$)

$$\sum_{k=1}^m \log|1 - e^{2\pi i a_k\theta}| \leq -\sum_{j=1}^J \sum_{k=1}^m \frac{\rho^j}{j}\cos 2\pi j a_k\theta + O\Big(\frac{m}{\sqrt{J}}\Big) \tag{2.6}$$

and we take J at least n to bound the last term in the right hand side of (2.5) by $\sqrt{n}$. We analyze the first term. Inserting (2.5) gives the sum of the following two expressions ((2.7) and (2.8))

$$-\sum_{j=1}^J \frac{\rho^j}{j}\Big(\frac{1}{2}F_n(j\theta) - \frac{1}{2}\Big) \tag{2.7}$$

$$-\sum_{j=1}^J \sum_{\ell=1}^n \frac{\rho^j}{j}(\xi_\ell - \mathbb{E}[\xi_\ell])\cos 2\pi\ell j\theta. \tag{2.8}$$

Since $F_n(j\theta) \geq 0$, (2.7) $\leq \log J$.

Rewrite

$$(2.8) = -\sum_{\ell=1}^n (\xi_\ell - \mathbb{E}[\xi_\ell])\Big[\sum_{j=1}^J \frac{\rho^j}{j}\cos 2\pi j\ell\theta\Big]. \tag{2.9}$$

Note that all frequencies in (2.9) are bounded by nJ.

Applying the probabilistic Salem-Zygmund inequality [Kol3] shows that with large probability

$$(2.9) \lesssim \sqrt{\log\, nJ}\Big[\sum_{\ell=1}^n \Big|\sum_{j=1}^J \frac{\rho^j}{j}\cos 2\pi j\ell\theta\Big|^2\Big]^{\frac{1}{2}}. \tag{2.10}$$

Our next task is to evaluate the expression $\sum_{\ell=1}^n \Big|\sum_{j=1}^J \frac{\rho^j}{j}\cos 2\pi j\ell\theta\Big|^2$.

A first observation is that we can assume

$$\|\theta\| > \frac{1}{10n} \tag{2.11}$$

since otherwise

$$|1-e^{2\pi i a_k\theta}|\leq 2\pi a_k\|\theta\|<\frac{2\pi}{10}<1$$

for all $k=1,\dots,m$, and also the left hand side of (2.4) is bounded by 1.

Next, we note that (since $\rho=1-\frac{1}{\sqrt{J}}$)

$$\begin{aligned}\Big|\sum_{j=1}^{J}\frac{\rho^j}{j}\cos 2\pi j\ell\theta\Big| &\leq \big|\log|1-\rho e(\ell\theta)|\big|+\frac{\rho^J}{J(1-\rho)}\\ &< \big|\log|1-\rho e(\ell\theta)|\big|+1.\end{aligned}$$

Hence

$$\sum_{\ell=1}^{n}\Big|\sum_{j=1}^{J}\frac{\rho^j}{j}\cos 2\pi j\ell\theta\Big|^2\lesssim\sum_{\ell=1}^{n}\big|\log|1-\rho e(\ell\theta)|\big|^2+n. \tag{2.12}$$

Fix θ and for $1<R\lesssim\log J$ define the dyadic set

$$S_R=\{1\leq\ell\leq n:\big|\log|1-\rho e(\ell\theta)|\big|\sim R\}.$$

Thus for $\ell\in S_R$

$$\|\ell\theta\|<|1-\rho e(\ell\theta)|<e^{-cR}=:\varepsilon.$$

Let $q\in\mathbb{N}$ be the smallest integer with $\|q\theta\|<2\varepsilon$. It follows that $|S_R|\lesssim\frac{n}{q}+1$. Assuming $q>R^3$, one obtains

$$\sum_{\ell\in S_R}\big|\log|1-\rho e(\ell\theta)|\big|^2\lesssim\Big(\frac{n}{R^3}+1\Big)R^2$$

with collected contribution (summing over dyadic R)

$$\sim n+(\log J)^2. \tag{2.13}$$

It remains to consider θ's with the property that for some large R and $q<R^3$,

$$\|q\theta\|<e^{-cR}.$$

Hence either θ admits a rational approximation

$$\Big|\theta-\frac{a}{q}\Big|<\frac{e^{-cR}}{q}<e^{-cR},\ \ q<R^3\ \text{ and }\ (a,q)=1 \tag{2.14}$$

or $\big($in (2.14) when $a=0\big)$, by (2.11)

$$\frac{1}{n}\lesssim\|\theta\|<e^{-cR}. \tag{2.15}$$

Consider first the case (2.15). Then

$$|S_R| \leq |\{\ell = 1, \dots, n : \|\ell\theta\| < e^{-cR}\}| \lesssim ne^{-cR}$$

and the above estimate still holds.

Assume next that θ satisfies (2.14). Write

$$\theta = \frac{a}{q} + \psi \text{ with } \beta = |\psi| < e^{-cR}. \tag{2.16}$$

First, we consider the case $\beta \gtrsim \frac{1}{nq}$.

Let $V \subset \{1, \dots, n\}$ be an interval of size $\sim \frac{1}{q\beta}$ so that $\{\ell\theta : \ell \in V\}$ consists of $q\beta$-separated points filling a fraction of $[0, 1]$ (mod 1). Hence

$$\begin{aligned}\sum_{\ell \in V} \big|\log|1 - \rho e(\ell\theta)|\big|^2 &\lesssim \frac{1}{\beta q} \int_0^1 \big|\log|1 - \rho e(t)|\big|^2 dt + \log^2(1 - \rho) \\ &\lesssim \frac{1}{\beta q} + \log^2 J\end{aligned}$$

and

$$\sum_{\ell=1}^{n} \big|\log|1 - \rho e(\ell\theta)|\big|^2 \lesssim n + nq\,\beta\log^2 n \lesssim n$$

unless

$$q\beta\log^2 n > 1, \text{ i.e. } \log n > e^{cR} \text{ or } R \lesssim \log\log n$$

where we used (2.14). Thus if $\beta \gtrsim \frac{1}{nq}$, (2.12) $\lesssim n(\log\log n)^2$.

The next case is $\beta < \frac{1}{100nq}$.

It follows that for $1 \leq \ell \leq n$

$$\Big|\ell\theta - \frac{\ell a}{q}\Big| < \frac{1}{100q}. \tag{2.17}$$

We obtain

$$\sum_{q \nmid \ell} \big|\log|1 - \rho e(\ell\theta)|\big|^2 \lesssim n \int_0^1 \big|\log|1 - \rho e(t)|\big|^2 dt \lesssim n$$

and

$$\sum_{q|\ell}\big|\log|1-\rho e(\ell\theta)|\big|^2 \sim \frac{1}{q\beta}\int_0^{n\beta}\big|\log|1-\rho e(t)|\big|^2 dt$$
$$\leq \frac{1}{q\beta}\int_0^{n\beta}\Big(\log\frac{1}{t}\Big)^2 dt \tag{2.18}$$
$$\lesssim \frac{n}{q}(\log n\beta)^2.$$

We obtain again a bound $O(n)$ unless

$$|\log n\beta| > \sqrt{q}$$

i.e.

$$\beta < \frac{e^{-\sqrt{q}}}{n}. \tag{2.19}$$

Thus (2.17) may be replaced by

$$\Big|\ell\theta - \ell\frac{a}{q}\Big| < e^{-\sqrt{q}} \text{ for } 1 \leq \ell \leq n. \tag{2.20}$$

For θ satisfying (2.20) we proceed in a different way. Write

$$\prod|1-e(a_k\theta)| = \prod_{j=1}^{n}|1-e(j\theta)|^{\xi_j}$$
$$\lesssim \prod_{j=1}^{n}\bigg(\Big|1-e\Big(j\frac{a}{q}\Big)\Big| + \frac{1}{q^{10}}\bigg)^{\xi_j}. \tag{2.21}$$

We replace ξ_j by its expectation $\mathbb{E}[\xi_j] = 1 - \frac{j}{n}$ using again a random argument. Thus if

$$\prod_{j=1}^{n}\bigg(\Big|1-e\Big(j\frac{a}{q}\Big)\Big| + \frac{1}{q^{10}}\bigg)^{1-\frac{j}{n}} \tag{2.22}$$

we have

$$|\log(2.21) - \log(2.22)| \leq \bigg|\sum_{j=1}^{n}\Big(\xi_j - \mathbb{E}[\xi_j]\Big)\log\bigg(\Big|1-e\Big(j\frac{a}{q}\Big)\Big| + \frac{1}{q^{10}}\bigg)\bigg|. \tag{2.23}$$

Recall that $q < R^3 \lesssim (\log J)^3 \sim (\log n)^3$. Thus with high probability we may bound (2.23) by $c\sqrt{n}\,\sqrt{\log\log n}\log q < c\sqrt{n}(\log\log n)^3$.

Hence

$$(2.21) \;\leq\; e^{c\sqrt{n}(\log\log n)^3}(2.22).$$

Partition $\{1, \ldots, n\}$ in intervals $I = [rq, (r+1)q - 1]$ and estimate for each such interval

$$\begin{aligned}
&\prod_{j\in I} \Big(\Big|1 - e\Big(j\frac{a}{q}\Big)\Big| + \frac{1}{q^{10}}\Big)^{1-\frac{j}{n}} \\
&\leq q^{c\frac{q^2}{n}} \Big[\frac{1}{q^{10}} \prod_{s=1}^{q-1} \Big(\Big|1 - e\Big(s\frac{a}{q}\Big)\Big| + \frac{1}{q^{10}}\Big]^{1-\frac{rq}{n}} \qquad (2.24)\\
&\leq q^{c\frac{q^2}{n}} \Big[\frac{1}{q^{10}} \prod_{s=1}^{q-1} \Big|1 - e\Big(\frac{s}{q}\Big)\Big|\Big]^{1-\frac{rq}{n}}.
\end{aligned}$$

The product $\prod_{s=1}^{q-1} \big|1 - e\big(\frac{s}{q}\big)\big|$ may be evaluated using Lemma 2.1 taking $J = q^2, \rho = 1 - \frac{1}{q}$. Thus clearly

$$\begin{aligned}
\sum_{s=1}^{q-1} \log\Big|1 - e\Big(\frac{s}{q}\Big)\Big| &\leq -\sum_{j=1}^{J} \frac{\rho^j}{j} \sum_{s=1}^{q-1} \cos 2\pi j\frac{s}{q} + O(1)\\
&\leq \sum_{\substack{1\leq j\leq J\\ q\nmid j}} \frac{\rho^j}{j} + q \sum_{\substack{1\leq j\leq J\\ q\mid j}} \frac{\rho^j}{j} + O(1)\\
&< \log q + C
\end{aligned}$$

implying that

$$(2.24) < q^{c\frac{q^2}{n}} \Big(\frac{1}{q^{10}} e^{\log q + c}\Big)^{1-\frac{rq}{n}} < q^{c\frac{q^2}{n}}. \qquad (2.25)$$

Since (2.22) is obtained as product of (2.24), (2.25) over the intervals I, we showed that

$$(2.22) < q^{c\frac{q^2}{n}n} 2^q < e^{(\log n)^3}.$$

Thus the preceding shows that if θ satisfies (2.20), then

$$\prod |1 - e(a_k\theta)| < e^{c\sqrt{n}(\log\log n)^3}. \qquad (2.26)$$

Going back to (2.10), omitting the case (2.20) estimated by (2.26), we obtained the bound $cn(\log\log n)^2$ on (2.12) which permits to majorize (2.8) by $c\sqrt{n\log n}(\log\log n)$ and $\prod |1 - e(a_k\theta)|$ by $e^{c\sqrt{n\log n}\log\log n}$. This completes the proof of Proposition 2.2.

3. **Almost full proportion**

It was observed in [E-S] that

$$\lim_{n\to\infty} M(1,\dots,n)^{\frac{1}{n}} \tag{3.1}$$

exists and lies strictly between 1 and 2.

This fact is in contrast with Proposition 2.2 which gives a subset $S\subset\{1,\dots,n\}, |S|\asymp\frac{n}{2}$ s.t.

$$\log M(S)\lesssim\sqrt{n}(\log n)^{\frac{1}{2}}\log\log n. \tag{3.2}$$

However

Proposition 3.1. *There is a constant $\tau>0$ such that if $S\subset\{1,\dots,n\}$ satisfies $|S|>(1-\tau)n$, then*

$$\log M(S)>cn \tag{3.3}$$

for some $c>0$.

Thus (3.3) generalizes (3.1) in some sense, but in view of (3.2), it fails dramatically if we do not assume $1-\frac{|S|}{n}$ small enough.

Proof of Proposition 3.1.

It will be convenient to use Fact 2 for an appropriate μ-convolution, which allow us to estimate the tail contribution in the k-summation.

Thus consider

$$-\min_\theta\Big\{\sum_{j\in S}\sum_{k=1}^\infty\frac{\cos 2\pi kj\cdot}{k}*\mu\Big\}(\theta)$$

$$=-\min_\theta\sum_{k=1}^\infty\sum_{j\in S}\frac{\hat\mu(jk)}{k}\cos 2\pi kj\theta$$

$$\geq-\min_\theta\sum_{k=1}^{k_0}\sum_{j=1}^{n}\frac{\hat\mu(jk)}{k}\cos 2\pi kj\theta \tag{3.4}$$

$$-(\log k_0)\pi n$$

$$-\sum_{k>k_0}\sum_{j=1}^{n}\frac{|\hat\mu(jk)|}{k} \tag{3.5}$$

since we assumed $|S| > (1-\tau)n$.

Separating in (3.4) the cases $k = 1$, and $2 \le k \le k_0$, we write

$$\begin{aligned}(3.4) \ge -\Big(\sum_{j=1}^{n}\cos 2\pi j\theta\Big) - \sum_{j=1}^{n}|1-\hat\mu(j)| \\ -\sum_{k=2}^{k_0}\frac{1}{k}\Big|\sum_{j=1}^{n}\hat\mu(jk)\cos 2\pi kj\theta\Big|.\end{aligned} \tag{3.6}$$

Take $\mu = F_{nR}(\theta)$, $R > 1$ an appropriate constant and $F_{nR}(\theta)$ the Féjer kernel.

Thus

$$\begin{aligned}\widehat{F}_{nR}(s) &= 1 - \frac{|s|}{nR} \quad \text{for } \ |s| \le nR \\ &= 0 \qquad\qquad \text{otherwise}.\end{aligned}$$

Take $\theta = \frac{3}{4n}$. The first term in (3.6) becomes, since

$$\sum_{j=1}^{n}\cos jx = \frac{1}{2}D_n(x) - \frac{1}{2}, \quad \text{where } \ D_n(x) = \frac{\sin(n+\frac{1}{2})x}{\sin\frac{x}{2}}$$

is the Dirichlet kernel,

$$\frac{1}{2} - \frac{1}{2}\frac{\sin\frac{3\pi}{2n}(n+\frac{1}{2})}{\sin\frac{3\pi}{4n}} \sim +\frac{1}{2\sin\frac{3\pi}{4n}}.$$

The second term is

$$-\sum_{j=1}^{n}\frac{j}{nR} = -\frac{n+1}{2R}.$$

The third term becomes

$$-\sum_{k=2}^{k_0}\frac{1}{k}\Big|\sum_{j=1}^{n}\Big(1-\frac{jk}{nR}\Big)_+\cos\pi\frac{3kj}{2n}\Big|. \tag{3.7}$$

By partial summation, the inner sum is bounded by

$$\begin{aligned}&\max_{j_1\le\min(n,\frac{nR}{k})}\Big|\sum_{j=1}^{j_1}\cos\pi\frac{3kj}{2n}\Big| \\ &= \max_{j_1\le\min(n,\frac{nR}{k})}\Big|\frac{1}{2}D_{j_1}\Big(\frac{3}{2}\pi\frac{k}{n}\Big) - \frac{1}{2}\Big| \\ &\le \frac{1}{2|\sin\frac{3}{4}\pi\frac{k}{n}|} + \frac{1}{2}.\end{aligned}$$

For $k < k_0 = o(n)$, the first term

$$\sim \frac{1}{2k \sin \frac{3\pi}{4n}}.$$

Hence

$$\begin{aligned}(3.7) &\geq -\sum_{k=2}^{k_0} \frac{1}{2k^2} \frac{1}{\sin \frac{3\pi}{4n}} - \log k_0 \\ &\geq -\frac{1}{2 \sin \frac{3\pi}{4n}} \Big(\frac{\pi^2}{6} - 1 \Big) - \log k_0.\end{aligned}$$

It follows from the preceding that

$$\begin{aligned}(3.4) &\geq +\frac{1}{2 \sin \frac{3\pi}{4n}} \Big(2 - \frac{\pi^2}{6} \Big) - \log k_0 - \frac{n+1}{2R} \\ &= cn - \log k_0\end{aligned}$$

for R a sufficiently large constant.

We bound (3.5) by

$$(3.5) \geq -\sum_{k \geq k_0} \frac{1}{k} \sum_{j \leq \frac{nR}{k}} 1 \geq -\sum_{k \geq k_0} \frac{nR}{k^2} \geq -\frac{R}{k_0} n.$$

In summary, we proved that

$$-\sum_{j \in S} \sum_{k=1}^{\infty} \frac{\hat{\mu}(jk)}{k} \cos 2\pi jk \frac{3}{4n} \geq cn - \log k_0 - \tau(\log k_0)n - \frac{C'n}{k_0} > \frac{c}{2} n$$

be choosing first k_0 large enough and then assuming τ sufficiently small.

This proves Proposition 3.1.

4. **Sets with large arithmetical Diameter**

As we pointed out the general lower bound $M(a_1, \ldots, a_n) > \sqrt{n}$ remains unimproved. However Proposition 4.1 stated below shows that in certain cases one can do better.

First, we give the following definition.

Definition. $D = \{v_1, \ldots, v_m\} \subset \mathbb{Z}$ *is called dissociated provided the relation*

$$\varepsilon_1 v_1 + \cdots + \varepsilon_m v_m = 0 \quad \textit{with} \;\; \varepsilon_i = 0, 1, -1$$

implies that $\varepsilon_1 = \cdots = \varepsilon_m = 0$.

We note that Hadamard lacunary sets are dissociated.

Proposition 4.1. *Assume* $S = \{a, \dots, a_n\}$ *contains a dissociated set* D *of size* m*. Then*

$$\log M(a_1, \dots, a_n) \gg \frac{m^{\frac{1}{2}-o(1)}}{(\log n)^{\frac{1}{2}}}. \tag{4.1}$$

Thus (4.1) improves the general lower bound from [E-S] provided $m > (\log n)^{3+\varepsilon}$.

Remark. By a result of Pisier [P], our assumption is equivalent to S containing a Sidon set Λ of size $|\Lambda| \sim m$. Here 'Sidon set' is in the harmonic analysis sense i.e.

$$\Big\| \sum_{n \in \Lambda} \lambda_n e(n\theta) \Big\|_\infty \geq c \sum |\lambda_n| \text{ for all scalars } \{\lambda_n\}$$

with $c = c(\Lambda)$ to be considered as a constant. (This concept is different from the Sidon sets in combinatorics!).

Dissociated sets are Sidon and conversely, Pisier proved that if Λ is a finite Sidon set, then Λ contains a proportional dissociated set.

Proof of Proposition 4.1.

We derive (4.1) from the equivalent statement

$$\max_\theta \big(\log |1 - e(a_1\theta)| + \dots + \log |1 - e(a_n\theta)| \big) \gg \frac{m^{\frac{1}{2}-o(1)}}{(\log n)^{1/2}} \tag{4.2}$$

which, since $\int \log |1 - e(a\theta)| = 0$ for $a \in \mathbb{Z} \backslash \{0\}$, is a consequence of the stronger claim that

$$\|F\|_1 \gg \frac{m^{\frac{1}{2}-o(1)}}{(\log n)^{1/2}} \tag{4.3}$$

denoting

$$F(\theta) = \log |1 - e(a_1\theta)| + \dots + \log |1 - e(a_n\theta)|.$$

Recall that by Fact 1

$$F(\theta) = -\sum_{k=1}^{\infty} \frac{1}{k} f(k\theta) \tag{4.4}$$

with

$$f(\theta) = \sum_{j=1}^{n} \cos(2\pi a_j \theta).$$

We first perform a finite Mobius inversion on (4.4). Recall that

$$\sum_{\substack{d|k, d\le r\\ d \text{ square free}}} \mu(d) = \begin{cases} 1 & \text{if } k=1 \\ 0 & \text{if } 1<k\le r \end{cases}$$

Hence

$$\begin{aligned} \sum_{\substack{d<r\\ \text{square free}}} F(d\theta)\frac{\mu(d)}{d} &= -\sum_{j=1}^{n}\sum_{k=1}^{\infty}\sum_{\substack{d<r\\ \text{square free}}} \cos(2\pi a_j dk\theta)\frac{\mu(d)}{dk} \\ &= -\sum_{j=1}^{n}\sum_{\ell=1}^{\infty}\frac{\cos(2\pi a_j\ell\theta)}{\ell}\Big[\sum_{\substack{d|\ell, d<r\\ \text{square free}}}\mu(d)\Big] \\ &= -f(\theta) - \sum_{j=1}^{n}\sum_{\ell>r}\frac{\cos(2\pi a_j\ell\theta)}{\ell}\Big[\sum_{\substack{d|\ell, d<r\\ \text{square free}}}\mu(d)\Big] \\ &= -f(\theta)+G(\theta), \end{aligned} \tag{4.5}$$

where

$$G(\theta) = -\sum_{j=1}^{n}\sum_{\ell>r}\frac{\cos(2\pi a_j\ell\theta)}{\ell}\Big[\sum_{\substack{d|\ell, d<r\\ \text{square free}}}\mu(d)\Big].$$

Note also that

$$\Big|\sum_{\substack{d|\ell, d<r\\ \text{square free}}}\mu(d)\Big| \le 2^{\omega(\ell)}, \tag{4.6}$$

where $\omega(\ell)$ is the number of distinct prime factors of ℓ.

Denote m the size of the largest dissociated set contained in $\{a_1, \dots, a_n\}$. Our first task will be to bound the Fourier transform $\|\hat{G}\|_\infty$ of G.

Thus given $t \in \mathbb{Z}$, we have

$$|\hat{G}(t)| \le \frac{1}{2}\sum_{j=1}^{n}\frac{a_j}{t}2^{\omega(\frac{t}{a_j})}. \tag{4.7}$$

We will bound (4.7) by considering dyadic ranges, letting for $K>r$ dyadic

$$J = J_K = \{j \in [1,n] \ : \ a_j|t \quad \text{and} \quad \frac{t}{a_j} \sim K\}.$$

Thus

$$\sum_{j\in J}\frac{a_j}{t}2^{\omega(\frac{t}{a_j})}\le\sqrt{\sum_{j\in J}\Big(\frac{a_j}{t}\Big)^2}\,\Big(\sum_{k\le K}4^{\omega(k)}\Big)^{\frac12}$$
$$\lesssim |J|^{\frac12}K^{-1}K^{\frac12}(\log K)^2=\Big(\frac{|J|}{K}\Big)^{\frac12}(\log K)^2. \tag{4.8}$$

Assume

$$|J|>\frac{K}{(\log K)^8}. \tag{4.9}$$

Our aim is to get a contradiction for appropriate choice of r.

At this point, we invoke the following result from [H-T] (see Fq (1.14)).

Denote

$$\psi(x,y)=\big|\{n\le x:\ \text{if }\ p|n,\ \text{then } p\le y\}\big|.$$

Lemma 4.2. *For any* $0<\alpha<1$, *we have*

$$\psi\big(x,(\log x)^{1/\alpha}\big)<x^{1-\alpha+o(1)}\ \textit{for}\ \ x\to\infty. \tag{4.10}$$

It follows from (4.9) that for any fixed $1>\alpha>0$, we have

$$|J|>2\psi\big(K,(\log K)^{\frac1\alpha}\big). \tag{4.11}$$

We make the following construction.

By (4.11), there is $j_1\in J$ such that $\frac{t}{a_{j_1}}$ has a prime divisor $p_1>(\log K)^{\frac1\alpha}$ and we write $\frac{t}{a_{j_1}}=p_1b_1$.

Next, let $J_1=\{j\in J: p_1|\frac{t}{a_j}\ \}$. Hence $|J_1|<\frac{K}{p_1}+1<\frac{K}{(\log K)^{\frac1\alpha}}<\frac{|J|}{(\log K)^{\frac1\alpha-8}}$ where we assume α taken much smaller than $\frac18$.

It follows that also

$$|J\backslash J_1|>\Big(2-\frac{1}{(\log K)^{\frac1\alpha-8}}\Big)\psi\big(K,(\log K)^{\frac1\alpha}\big)$$

which permits to introduce $j_2\in J\backslash J_1$ and a prime $p_2>(\log K)^{\frac1\alpha}$ such that $p_2|\frac{t}{a_{j_2}}$. Write $\frac{t}{a_{j_2}}=p_2b_2$. Clearly $p_2\ne p_1$ and $p_1\nmid b_2$.

The contribution of the process is clear. We may introduce elements

$$j_1,\dots,j_s\in J\ \text{ with }\ s\gtrsim(\log K)^{\frac1\alpha-8}$$

and prime divisors $p_{s'}|\frac{t}{a_{j_{s'}}}$. Write $\frac{t}{a_{j_{s'}}} = p_{s'}b_{s'}$ such that $p_{s'} \nmid \frac{t}{a_{j_{s''}}}$ for $s' < s''$. Hence $p_{s''} \neq p_{s'}$ for $s' \neq s''$ and

$$p_{s'} \nmid b_{s''} \text{ for } s' < s''. \tag{4.12}$$

We claim that the set $\{a_{j_1}, \dots, a_{j_s}\}$ is dissociated. Otherwise, there is a non-trivial relation

$$\varepsilon_1 a_{j_1} + \cdots + \varepsilon_s a_{j_s} = 0 \qquad \text{with } \varepsilon_{s'} = 0, 1, -1$$

which by the preceding translates in

$$\varepsilon_1 \frac{1}{p_1 b_1} + \cdots + \varepsilon_s \frac{1}{p_s b_s} = 0$$

or

$$\sum_{s'=1}^{s} \varepsilon_{s'} \prod_{s'' \neq s'} p_{s''} b_{s''} = 0.$$

Let s_1 be the smallest s' with $\varepsilon_{s'} \neq 0$. Then

$$\sum_{s'=s_1}^{s} \varepsilon_{s'} \prod_{\substack{s'' \neq s' \\ s'' \geq s_1}} p_{s''} b_{s''} = 0. \tag{4.13}$$

Since

$$p_{s_1} \Big| \prod_{\substack{s'' \neq s' \\ s'' \geq s_1}} p_{s''} b_{s''} \text{ for } s' > s_1,$$

identity (4.13) implies

$$p_{s_1} \Big| \prod_{s'' > s_1} b_{s''},$$

contradicting (4.12).

Hence $\{a_{j_1}, \dots, a_{j_s}\}$ is dissociated and by definition of m,

$$s \leq m$$

implying

$$m \geq (\log K)^{\frac{1}{\alpha} - 8} \quad \text{and} \quad \log r \leq m^{\frac{\alpha}{1-8\alpha}}.$$

Thus, by taking

$$\log r \sim m^{2\alpha} \quad (\alpha \text{ small enough})$$

we obtain a contradiction under assumption (4.9).

Hence

$$|J_K| < \frac{K}{(\log K)^8} \quad \text{for } K > r$$

and summing (4.8) over dyadic ranges of $K > r$ gives the bound

$$|\hat{G}(t)| < \sum_{\substack{K>r \\ \text{dyadic}}} \frac{1}{(\log K)^2} \lesssim \frac{1}{\log r}. \tag{4.14}$$

Consequently

$$\widehat{(4.5)}(t) = -\hat{f}(t) + O\Big(\frac{1}{\log r}\Big) = -\hat{f}(t) + o(1) \text{ for all } t \in \mathbb{Z}. \tag{4.15}$$

Since

$$\hat{f}(j) = \frac{1}{2},$$

we have

$$\widehat{(4.5)}(j) = -\frac{1}{2} + o(1). \tag{4.16}$$

Next, let D be a size m dissociated set in $\{a_1, \dots, a_n\}$. Define

$$\varphi(\theta) = \frac{1}{\sqrt{m}} \sum_{j \in D} e(j\theta).$$

Also, let Φ, Ψ be the dual Orliez functions

$$\Phi(x) = x\sqrt{\log(2+x)} \quad \text{and} \quad \Psi(x) = e^{x^2}.$$

It is well known (e.g. Theorem 3.1 in [Rud].) that

$$\|\varphi\|_{L^\Psi} = O(1).$$

By (4.16)

$$\Big(\frac{1}{2} - o(1)\Big)\sqrt{m} \le \Big| \int_0^1 (4.5)\varphi(\theta) d\theta \Big| \le C\|(4.5)\|_{L^\Phi} \tag{4.17}$$

It remains to bound $\|(4.5)\|_{L^\Phi}$.

Estimate

$$\begin{aligned} &\int |(4.5)| \sqrt{\log(|(4.5)|+2)}\, d\theta \\ \le \ &\sum_{j>0} 2^{j/2} \int_{2^{2^{j-1}} \le \lambda \le 2^{2^j}} \mu(M)\, d\lambda, \end{aligned} \tag{4.18}$$

Where $M = \{\theta : (4.5)(\theta) > \lambda\}$ and μ is the measure. Using the left hand side of (4.5), the j-summands is bounded by

$$2^{j/2}\|(4.5)\|_1 \lesssim 2^{j/2}\log r\ \|F\|_1. \tag{4.19}$$

Also, let $\Psi_1(u) = e^u$. Then

$$\Big\|\sum_{d\le r}\frac{|F(d\theta)|}{d}\Big\|_{L^{\Psi_1}} \le (\log r)\|F\|_{L^{\Psi_1}} \lesssim n\log r,$$

since $\|\log|1-e^{i\theta}|\,\|_{L^{\Psi_1}} < \infty$.

Thus also the bound

$$\mu(M) \le e^{-c\,\frac{\lambda}{n\log r}}$$

implying the following bound for the j-summands

$$2^{j/2}2^{2^j}e^{-c\,\frac{2^{2^{j-1}}}{n\log r}}. \tag{4.20}$$

Hence

$$(4.18) < \sum_j 2^{j/2}\min\Big((\log r)\|F\|_1, 2^{2^j}e^{-c\,\frac{2^{2^{j-1}}}{n\log r}}\Big).$$

For $2^{2^{j-2}} < n\log r$, we get the contribution

$$(\log n)^{\frac12}\log r\|F\|_1.$$

For $2^{2^{j-2}} \ge n\log r$, we bound by

$$(n\log r)^{4+\epsilon}e^{-cn\log r} + (n\log r)^{4\cdot2+\epsilon}\,e^{-c(n(\log r))^3} + \cdots + (n\log r)^{4\cdot2^{u-1}+\epsilon}\,e^{-c(n\log r)^{2^u-1}} + \cdots$$
$$< O(1).$$

Hence

$$\|(4.5)\|_{L^\Phi} \lesssim (4.18) < (\log n)^{\frac12}m^{2\alpha}\|F\|_1 \tag{4.21}$$

recalling above choice for $\log r$.

Returning to (4.17), we proved that

$$\Big(\frac12 - o(1)\Big)m^{\frac12-2\alpha} \lesssim (\log n)^{\frac12}\|F\|_1$$

hence

$$\|F\|_1 \gtrsim m^{\frac12-\varepsilon}(\log n)^{-\frac12}.$$

This proves (4.3) and hence Proposition 4.1.

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