A Remark on Sieving in Biased Coin Convolutions *†

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Abstract

In this work, we establish a nontrivial level of distribution for densities on $\{1, \dots, N\}$ obtained by a biased coin convolution. As a consequence of sieving theory, one then derives the expected lower bound for the weight of such densities on sets of almost primes.

Introduction.

Over the recent years, there has been an increasing interest in sieving problems in combinatorial objects without a simple arithmetic structure. The typical example is that of finitely generated 'thin subgroups' of linear groups such as $SL_2(\mathbb{Z})$ or $SL_2(\mathbb{Z} + i\mathbb{Z})$. These groups are combinatorially defined but are not arithmetic (they are of infinite index) and as such cannot be studied with classical automorphic techniques. Examples of natural appearances of this type of questions include the study of the curvatures in integral Apollonian circle packings, Pythagorean triples and issues around fundamental discriminates of quadratic number fields and low lying geodesics in the modular surface. (See [2].) The reader may also wish to consult the excellent Bourbaki exposition by E. Kowalski [6] for a detailed account of many of these recent developments around 'exotic sieving'.

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In this paper we consider a slightly different problem but in a somewhat similar spirit. Let $N = 2^m$ and identify $\{1, \dots, N\}$ with the Boolean cube $\{0,1\}^m$ through binary expansion. Denote μ_ρ the probability measure on $\{0,1\}^m$ given by a standard biased coin convolution, i.e. on each factor we take an independent distribution assigning probability ρ to 0 and $1-\rho$ to 1. Consider the resulting distribution on $\{1, \dots, N\}$. For $\rho = \frac{1}{2}$, this is the uniform distribution while for $\rho \to 1$, these distributions become increasingly singular. Our aim is to study some of their arithmetical properties and in particular prove that there is a nontrivial level of distribution no matter how close ρ is to 1, $\rho < 1$. Similar results may also be obtained for g-adic analogues, expanding integers in base g.

Notations.

Throughout the paper, any implied constants in the symbols O, $\lesssim$, $\gtrsim$ and $\sim$ may depend on the parameter $\rho > 0$. We recall that the notations $A = O(B)$, $A \lesssim B$ and $B \gtrsim A$ are all equivalent to the statement that the inequality $|A| \leq cB$ holds with some constant $c > 0$, and that $A \sim B$ means A is equal to B asymptotically.
The distance from x to the nearest integer is denoted by $\|x\|$.
For brevity, we set

$$e(\theta) = e^{2\pi i\theta}, \quad e_q(\theta) = e\left(\frac{\theta}{q}\right).$$

The constant c may vary in the same statement and depend on the parameter ρ.

1 The statement.

Consider the distribution μ on $[1, N] \cap \mathbb{Z}$, with $N = 2^m$, induced by the random variable $\sum_j \xi_j 2^j$ with $(\xi_j), j \geq 0$, be an independent, identically distributed sequence of random variables taking values in $\{0,1\}$, $\mathbb{P}(\xi_j = 0) = \rho$, $\mathbb{P}(\xi_j = 1) = 1-\rho$, $\frac{1}{2} < \rho < 1$. Thus, if $n = \sum_j a_j 2^j$ with $a_j \in \{0,1\}$ the binary expansion, then

$$\mu(n) = \rho^{m-\ell}(1-\rho)^\ell, \quad \text{where } \ell = \sum_j a_j \tag{1.1}$$

Note that for $\rho = \frac{1}{2}$ we obtain the normalized uniform measure on $[0, N]$.

The measure (1.1) has dimension $(1-\rho)\log\frac{1}{1-\rho}$ and hence becomes more irregular for $\rho \to 1$. Our aim is to establish a level of distribution of μ in the sense of sieving theory. Thus, taking $q < N^{\alpha}$, q square free and α appropriately small, (since μ is normalized) we may write

$$\begin{aligned}\mu\big[n \le N : q|n\big] =& \frac{1}{q}\sum_{\lambda=0}^{q-1}\sum_{n=1}^{N} e_q(\lambda n)\mu(n) \\ =& \frac{1}{q} + R_q,\end{aligned} \tag{1.2}$$

where

$$R_q = \frac{1}{q}\sum_{\lambda=1}^{q-1}\sum_{n=1}^{N} e_q(\lambda n)\mu(n).$$

We also assume q odd. The number α is the *sieving exponent* [5].

Our aim is to obtain a bound of the form

$$\sum_{q<N^{\alpha}}{}' |R_q| = o\bigg(\frac{1}{\log N}\bigg) \tag{1.3}$$

where $\sum'$ sums over q square free and odd. (See (4.1) for a better bound.)

Theorem 1. *Let the notations be as above. Then μ has sieving exponent $\alpha(\rho) > 0$. In fact, $\alpha(\rho) = O(1-\rho)$ for $\rho \to 1$.*

Theorem 1 will be proved in Section 4.

Theorem 1 provides the necessary ingredient to apply sieving theory and in order to obtain *almost primes* (integer whose prime factors are bounded below) satisfying certain property. More precisely, from the standard combinatorial sieve, we also apply to measure instead of set. (See e.g. Corollary 6.2 in [4]. Also, [1], [2], [3].) We have the following result.

Corollary 2. *Let μ be the probability distribution defined as above, and let $\mathcal{P}_{N,\alpha} = \{n \le N :$ the prime factors of n are at least $N^{\frac{\alpha}{20}}\}$. Then there are constants $0.81 > C_1 > C_2 > 0.36$ such that for sufficiently large N we have*

$$\frac{C_2}{\alpha\log N} < \mu(\mathcal{P}_{N,\alpha}) < \frac{C_1}{\alpha\log N}. \tag{1.4}$$

(See (4.7) for C_1 and C_2.)

Remark. If $n \le N$ has no prime factors less than $N^{\frac{\alpha}{20}}$, then n has at most $[20/\alpha]$ prime factors. Since α is small, the number of prime factors in the almost primes is rather large.

2 First estimates.

Let

$$\begin{aligned} R_q =& \frac{1}{q}\sum_{\lambda=1}^{q-1}\sum_{n=1}^{N} e_q(\lambda n)\mu(n) \\ =& \frac{1}{q}\sum_{\lambda=1}^{q-1}\prod_{j<m}\left(\rho + (1-\rho)\, e\left(\frac{\lambda 2^j}{q}\right)\right). \end{aligned}$$

Therefore,

$$|R_q| \le \frac{1}{q}\sum_{\lambda=1}^{q-1}\prod_{j<m}\left|\rho + (1-\rho)\, e\left(\frac{\lambda 2^j}{q}\right)\right|. \tag{2.1}$$

Then

$$\begin{aligned} \sum_{q<N^\alpha}|R_q| &\le \sum_{q_2<N^\alpha}\sum_{q_1<N^\alpha}\frac{1}{q_1 q_2}\sum_{\substack{\lambda_1=1\\(q_1,\lambda_1)=1}}^{q_1-1}\prod_{j<m}\left|\rho + (1-\rho)\, e\left(\frac{\lambda_1 2^j}{q_1}\right)\right| \\ &\le \log N^\alpha \sum_{q<N^\alpha}\frac{1}{q}\sum_{\substack{\lambda=1\\(q,\lambda)=1}}^{q-1}\prod_{j<m}\left|\rho + (1-\rho)\, e\left(\frac{\lambda 2^j}{q}\right)\right|. \end{aligned} \tag{2.2}$$

Define

$$\widetilde{R_q} = \frac{1}{q}\sum_{\substack{\lambda=1\\(q,\lambda)=1}}^{q-1}\prod_{j<m}\left|\rho + (1-\rho)\, e\left(\frac{\lambda 2^j}{q}\right)\right|. \tag{2.3}$$

Note that

$$|\rho + (1-\rho)e(\theta)|^2 = 1 - 4\rho(1-\rho)\sin^2\pi\theta. \tag{2.4}$$

Let us consider first the case of small q.

For $\lambda \neq 0 \bmod q$, identity (2.4) implies

$$\left|\rho + (1-\rho)\, e\Big(\frac{\lambda 2^j}{q}\Big)\right| \leq 1 - \frac{c}{q^2}$$

for $c > 0$ so that $\widetilde{R_q} < \big(1 - \frac{c}{q^2}\big)^m < e^{-\frac{c}{q^2}m} < N^{-c/q^2}$.

One can do better by the following observation.

Let $I \subset \{1, \dots, m\}$ be an arbitrary interval of size $|I| = [\log q] + 1$ and let $\lambda \in \mathbb{Z}$ be an integer satisfying $\lambda \neq 0 \bmod q$.

Claim.

$$\max_{j \in I} \left|\sin\left(\frac{\lambda 2^j}{q}\pi\right)\right| > c, \tag{2.5}$$

where $c > 0$ is some constant independent of q.

Proof of Claim. Let $I = \{j_0, j_0 + 1, \dots, j_0 + s\}$ with $s = [\log q]$ and let

$$\xi = \left\|\frac{\lambda 2^{j_0}}{q}\right\|$$

The assumptions that $\lambda \neq 0 \bmod q$ and that q is odd imply $\xi \neq 0$, and hence $\xi \geq \frac{1}{q}$.

Fix a constant c sufficiently small. Assume $\xi < c$. It is clear that for $s' \in \mathbb{N}$ small, $\big\|\frac{\lambda 2^{j_0+s'}}{q}\big\| = 2\big\|\frac{\lambda 2^{j_0+s'-1}}{q}\big\| = 2^{s'}\xi$. It is also clear that $\big\|\frac{\lambda 2^{j_0+s'}}{q}\big\| > c$ for some $s' \leq s$. Otherwise, $2^s \leq \frac{c}{\xi} \leq cq$, which contradicts to the assumption $s \sim \log q$. Now, the claim follows from the fact that $|\sin(\theta\pi)| \sim \|\theta\|\pi$ for θ small. $\square$

Therefore, dividing the index interval $[1, m]$ in (2.3) in subintervals of length $\sim \log q$, by (2.4) and (2.5), we also have

$$\widetilde{R_q} < \big(1 - c(\rho)\big)^{\frac{m}{[\log q]+1}} < N^{-\frac{c(\rho)}{[\log q]+1}} < e^{-c(\rho)\sqrt{\log N}} \tag{2.6}$$

if $\log q = O\big(\sqrt{\log N}\big)$. (To obtain the first inequality, in each subinterval of j we use (2.5) for the factor with $\sin^2\big(\frac{\lambda 2^j}{q}\pi\big)$ achieving maximum, and the trivial bound 1 for the other factors.)

Remark. In (2.6) we can make $c(\rho)$ explicit by choosing s and c explicitly, and using the inequality $|\sin(\theta\pi)| > \frac{\|\theta\|\pi}{2}$ for $\|\theta\|$ small in the proof of the Claim. More precisely, if we take $s = [\log_2 q] + 1$ and $c = \frac{1}{10}$, we can take $c(\rho) = \rho(1-\rho)\pi/10$ in (2.6).

3 Further estimates.

We want to estimate

$$\sum_{Q\le q<2Q-1} \widetilde{R_q}$$

with $Q < N^{\alpha}$ and $\log Q \gtrsim \sqrt{\log N}$. We will show that

$$\sum_{Q\le q<2Q-1} \widetilde{R_q} \lesssim Q^{-\frac{1}{2}} \tag{3.1}$$

To show (3.1) we may assume $\alpha = \frac{1}{t}$ for some large even integer $t > 2$ $\big($given in (3.7)$\big)$. Choose $h \in \mathbb{Z}$ such that

$$2^h \le Q^2 < 2^{h+1}. \tag{3.2}$$

Since $Q < N^{\alpha}$, we have

$$h < \frac{2}{t}m < m.$$

Estimate (3.1) using Hölder inequality

$$\begin{aligned}
&\sum_{Q\le q<2Q-1} \widetilde{R_q} \\
\le\ & \sum_{Q\le q<2Q-1} \frac{1}{Q} \sum_{\substack{\lambda=1\\(q,\lambda)=1}}^{q-1} \prod_{\tau=1}^{t/2} \prod_{j=(\tau-1)h}^{\tau h-1} \left|\rho + (1-\rho)\, e\Big(\frac{\lambda 2^j}{q}\Big)\right| \\
\le\ & \sum_{Q\le q<2Q-1} \frac{1}{Q}\, q^{1-\frac{2}{t}} \Big[\sum_{\substack{\lambda=1\\(q,\lambda)=1}}^{q-1} \prod_{\tau=1}^{t/2} \prod_{j=(\tau-1)h}^{\tau h-1} \left|\rho + (1-\rho)\, e\Big(\frac{\lambda 2^j}{q}\Big)\right|^{t/2} \Big]^{2/t} \\
\lesssim\ & \sum_{Q\le q<2Q-1} \Big[\prod_{\tau=1}^{t/2} \frac{1}{Q} \sum_{\substack{\lambda=1\\(q,\lambda)=1}}^{q-1} \prod_{j=(\tau-1)h}^{\tau h-1} \left|\rho + (1-\rho)\, e\Big(\frac{\lambda 2^j}{q}\Big)\right|^{t/2} \Big]^{2/t} \\
\le\ & \prod_{\tau=1}^{t/2} \Big[\sum_{Q\le q<2Q-1} \frac{1}{Q} \sum_{\substack{\lambda=1\\(q,\lambda)=1}}^{q-1} \prod_{j=(\tau-1)h}^{\tau h-1} \left|\rho + (1-\rho)\, e\Big(\frac{\lambda 2^j}{q}\Big)\right|^{t/2} \Big]^{2/t} \\
\le\ & \prod_{\tau=1}^{t/2} \Big[\sum_{Q\le q<2Q-1} \frac{1}{Q} \sum_{\substack{\lambda=1\\(q,\lambda)=1}}^{q-1} \prod_{j=0}^{h-1} \left|\rho + (1-\rho)\, e\Big(\frac{\lambda 2^{(\tau-1)h} 2^j}{q}\Big)\right|^{t/2} \Big]^{2/t}
\end{aligned}$$

$$\leq \sum_{Q\leq q<2Q-1} \frac{1}{Q} \sum_{\substack{\lambda'=1\\(q,\lambda')=1}}^{q-1} \prod_{j=0}^{h-1} \left|\rho + (1-\rho)\, e\Big(\frac{\lambda' 2^j}{q}\Big)\right|^{t/2}. \tag{3.3}$$

To finish the estimate, we need the following two lemmas.

Lemma 3. *For all θ, $0 < \delta < e^{-\frac{1}{2}}$ and*

$$\ell > \frac{\log \frac{1}{\delta}}{\rho(1-\rho)}, \tag{3.4}$$

we have

$$|\rho + (1-\rho)e(\theta)|^{2\ell} \leq 1 - (1-\delta)\sin^2 \pi\theta. \tag{3.5}$$

Proof. Let

$$\gamma = 4\rho(1-\rho)\sin^2 \pi\theta.$$

By (2.4),

$$|\rho + (1-\rho)e(\theta)|^{2\ell} = (1-\gamma)^{\ell}.$$

We consider the following two cases.

(i). $\gamma > \frac{1}{\ell}\log\frac{1}{\delta}$.

Then

$$(1-\gamma)^{\ell} \leq e^{-\ell\gamma} < \delta < 1 - (1-\delta)\sin^2 \pi\theta.$$

(ii). $\gamma \leq \frac{1}{\ell}\log\frac{1}{\delta}$.

Let

$$\ell_1 = \frac{\ell}{2\log\frac{1}{\delta}} < \ell,$$

and estimate

$$\begin{aligned}(1-\gamma)^{\ell} < (1-\gamma)^{\ell_1} < e^{-\ell_1\gamma} &< 1 - \frac{1}{2}\ell_1\gamma \\ &= 1 - \frac{\ell\rho(1-\rho)}{\log\frac{1}{\delta}}\sin^2 \pi\theta \\ &< 1 - \sin^2 \pi\theta \\ &< 1 - (1-\delta)\sin^2 \pi\theta.\end{aligned}$$

(Note that the third inequality is because $\ell_1\gamma = \frac{\ell\gamma}{2\log\frac{1}{\delta}} \leq \frac{1}{2}$.) $\square$

Lemma 4. *Let γ be positive. Then for all θ and $0 < \delta < 1$, we have*

$$1 - (1-\delta)\sin^2\theta < 1 + \gamma - (1-\delta)\sin^2(\theta+\gamma). \tag{3.6}$$

Proof. Using the identity

$$\sin^2 A - \sin^2 B = \sin(A+B)\sin(A-B)$$

on the difference of both sides of (3.6), we obtain

$$\gamma - (1-\delta)\sin(2\theta+\gamma)\sin\gamma \geq \gamma - (1-\delta)\gamma > 0. \qquad \square$$

Let

$$t > \frac{4\log\frac{1}{\delta}}{\rho(1-\rho)}. \tag{3.7}$$

With $\theta = \lambda 2^j/q$, Lemma 3 implies that (3.3) is bounded by

$$\frac{1}{Q}\sum_{Q\leq q<2Q-1}\sum_{\substack{\lambda=1\\(q,\lambda)=1}}^{q-1}\prod_{j=0}^{h-1}\left(1-(1-\delta)\sin^2\left(\frac{\pi\lambda 2^j}{q}\right)\right). \tag{3.8}$$

First, we will replace the discrete sum $\sum_{Q\leq q<2Q-1}\ \sum_{\substack{1\leq\lambda<q\\(\lambda,q)=1}}$ by integral. For Q given, let

$$S = \left\{\frac{\lambda}{q} : 1\leq\lambda<q,\ \ Q\leq q<2Q, (\lambda,q)=1\right\}\subset[0,1].$$

We note that S is $\frac{1}{(2Q-2)(2Q-1)}$ separated. (We recall that a set S is H-separated if for all distinct $x, y\in S$, $|x-y|\geq H$.)

In Lemma 4, taking $\gamma = \pi 2^j\beta'$ with $\beta'\in[0,\beta]$ for some $\beta = O(2^{-h})$ to be specified later, we bound (3.8) by

$$\frac{1}{Q}\sum_{\frac{\lambda}{q}\in S}\prod_{j=0}^{h-1}\left(1+\gamma-(1-\delta)\sin^2\left(\pi 2^j(\frac{\lambda}{q}+\beta')\right)\right). \tag{3.9}$$

We will use integration to bound (3.9) by replacing S by $S_\beta = S + [0, \beta]$. Since S_β is disjoint union of subintervals in $[0, 1]$, averaging over $\beta' \in [0, \beta]$ gives

$$\frac{1}{\beta Q} \int_{S_\beta} \prod_{j=0}^{h-1} \big(1 + \gamma - (1 - \delta) \sin^2(\pi 2^j x)\big) dx$$

$$\lesssim \frac{1}{\beta Q} \int_0^1 \prod_{j=0}^{h-1} \big(1 + \gamma - (1 - \delta) \sin^2(\pi 2^j x)\big) dx. \tag{3.10}$$

More precisely, we take

$$\beta = \frac{\delta}{4}\, Q^{-2}, \tag{3.11}$$

(which implies $\gamma < \delta$) and bound (3.10) by

$$\begin{aligned} &\frac{4}{\delta}\, Q \int_0^1 \prod_{j=0}^{h-1} \big(1 + \delta - (1 - \delta) \sin^2(\pi 2^j x)\big) dx \\ &= \frac{4}{\delta}\, Q \bigg(1 + \delta - \frac{1 - \delta}{2}\bigg)^h \\ &= \frac{4}{\delta}\, Q \bigg(\frac{1 + 3\delta}{2}\bigg)^h \\ &\lesssim\ Q^{-1/2}, \end{aligned} \tag{3.12}$$

for δ small enough (such that $1 + 3\delta < \sqrt[4]{2}$).

Putting (3.3), (3.8)-(3.10) and (3.12) together, we obtain the intended bound on (3.1).

4 The proofs.

Proof of Theorem 1.

We will use the estimates (2.6) and (3.1) to prove (1.3). First, we will

bound $\sum_{q<N^\alpha} \widetilde{R_q}$.

$$\begin{aligned}\sum_{q<N^\alpha} \widetilde{R_q} =& \sum_{\log q<c_1\sqrt{\log N}} \widetilde{R_q} + \sum_{\log q\geq c_1\sqrt{\log N}} \widetilde{R_q} \\ <& e^{c_1\sqrt{\log N}}\, e^{-c\sqrt{\log N}} + \sum_{2^j\geq e^{c_1\sqrt{\log N}}} (2^j)^{-\frac{1}{2}} \\ =& e^{-c_1\sqrt{\log N}} + \frac{\left(e^{c_1\sqrt{\log N}}\right)^{-\frac{1}{2}}\left(1-(N^\alpha)^{-\frac{1}{2}}\right)}{1-2^{-\frac{1}{2}}} \\ \lesssim& e^{-\frac{c_1}{2}\sqrt{\log N}}.\end{aligned} \tag{4.1}$$

(Here we take c_1 sufficiently small, e.g. $c_1 = \frac{\rho(1-\rho)c}{c_1\log 2}$ with c given in (2.5).) From (2.2) and (2.3),

$$\begin{aligned}\sum_{q<N^\alpha}{}' |R_q| \leq \sum_{q<N^\alpha} |R_q| \leq& \log N^\alpha \sum_{q<N^\alpha} \widetilde{R_q} \\ \lesssim& \log N^\alpha e^{-\frac{c_1}{2}\sqrt{\log N}} < \ o\bigg(\frac{1}{\log N}\bigg),\end{aligned}$$

which goes to 0 as N goes to infinity.

From (3.7),

$$\alpha = \frac{1}{t} < \frac{\rho(1-\rho)}{4\log\frac{1}{\delta}}\,.$$

Hence $\alpha = O(1-\rho)$ for $\rho\to 1$. $\square$

Corollary 2 follows immediately from Theorem 1 and Corollary 6.2 in [4]. For the readers' convenience we will include the statement of the latter here. First, we introduce some notations.

Let $\{a_n : n\in\mathbb{N}\}\subset\mathbb{R}_{\geq 0}$,

$$P(z) = \prod_{p<z,\, p \text{ prime}} p\,, \qquad S(x,z) = \sum_{n\leq x,\, (n,P(z))=1} a_n$$

and

$$A_d(x) = \sum_{n\leq x,\, n\equiv 0 \mod d} a_n.$$

Assume

$$A_d(x) = \frac{1}{d}\sum_{n\leq x} a_n + r_d(x). \tag{4.2}$$

Then we denote

$$V(z) = \prod_{p|P(z)} \left(1 - \frac{1}{p}\right)$$

$$R(x,y) = \sum_{d<y,\ d|P(z)} |r_d(x)|, \ \ y < N^{\alpha}$$

Let $K > 1$ and $\kappa > 0$ satisfy

$$\prod_{w \le p < z} \left(1 - \frac{1}{p}\right)^{-1} \le K \left(\frac{\log z}{\log w}\right)^{\kappa} \tag{4.3}$$

and let $\beta = 9\kappa + 1$, any $s \ge \beta$.

The following is Corollary 6.2 in [4].

Proposition. [IK] *Let notations be as above. Then*

$$S(x,z) < (1 + e^{\beta - s} K^{10}) V(z) X + R(x, z^s) \tag{4.4}$$

$$S(x,z) > (1 - e^{\beta - s} K^{10}) V(z) X - R(x, z^s) \tag{4.5}$$

Proof of Corollary 2.

Since

$$\prod_{p<z} \left(1 - \frac{1}{p^2}\right) e^{-\sum_{p<z} \frac{1}{p}} < \prod_{p<z} \left(1 - \frac{1}{p}\right) < e^{-\sum_{p<z} \frac{1}{p}}$$

and

$$\lim_{z \to \infty} \left(\sum_{p<z} \frac{1}{p} - \log\log(z) \right) = 0.26\cdots, \quad \sum_{p=2}^{\infty} \log\left(1 - \frac{1}{p^2}\right) \approx -0.68816\cdots,$$

we have

$$\frac{1}{e^{0.95} \log z} < V(z) < \frac{1}{e^{0.26} \log z}. \tag{4.6}$$

So in (4.3), we may take $\kappa = 1$, $K = e^{0.68816} \approx 1.99$. Hence $\beta = 10, s = 20$. In the Proposition, we take $x = N$ and $z = N^{\frac{\alpha}{20}}$. Therefore, $R(N, N^{\alpha}) \le \sum_{q<N^{\alpha}} |R_q|$, and the constants in (4.4) and (4.5) are $(1 - (1.99/e)^{10})/e^{0.95} = 0.369635\cdots$ and $(1 + (1.99/e)^{10})/e^{0.26} = 0.805154\cdots$.

Hence we obtain the following bounds.

$$\frac{0.369635}{\alpha \log N} - \frac{o(1)}{\log N} < \mu(\mathcal{P}_{N,\alpha}) < \frac{0.805154}{\alpha \log N} + \frac{o(1)}{\log N}. \qquad \square \tag{4.7}$$

(We recall that $\alpha = \frac{1}{t}$ with $t = t(\rho)$ given in (3.7).)

5 Random polynomials with coefficients in $\{0, 1, -1\}$.

The initial motivation for this work came from [7], where one considers biased coin convolution densities for ternary expansions, with probabilities $\mathbb{P}(\xi = 0) = \rho_0$, $\mathbb{P}(\xi = 1) = \rho_1$, $\mathbb{P}(\xi = -1) = \rho_{-1}$ and $\rho_0 \geq \rho_1, \rho_{-1}$. The main problem focused in [7] is to ensure that the set of integers $\{n < N : q^2|n \text{ for some } q > Q\}$ carries small weight for $Q \to \infty$, which they manage to ensure if q is not too large. The natural problem is whether such restriction is necessary. Clearly, this issue may be rephrased as the sieving problem for square free integers, but with unrestricted level of distribution. (The large values of q are indeed the problematic ones.) While we are unable to provide a definite answer to their question and the main result of this note does not directly contribute, we will point out a simple probabilistic argument leading to the replacement of their condition. Our argument uses virtually no arithmetic structure.

Let $(\xi_j), j \geq 0$, be an independent, identically distributed sequence of random variables taking values in $\{-1, 0, 1\}$. Let $m \geq 1$ and define the random polynomial P by

$$P(z) := \sum_{j=0}^{m} \xi_j z^j.$$

In [7], the authors assumed that

$$\max_{x \in \{-1,0,1\}} \mathbb{P}(\xi_0 = x) < \frac{1}{\sqrt{3}} = 0.5773\ldots \tag{5.1}$$

and proved that $\mathbb{P}(P \text{ has a double root}) = \mathbb{P}(P \text{ has } -1, 0 \text{ or } 1 \text{ as a double root})$ up to a $o(m^{-2})$ factor, and $\lim_{m\to\infty} \mathbb{P}(P \text{ has a double root }) = \mathbb{P}(\xi_0 = 0)^2$. One of the open problems they raised at the end of the paper asked whether it is necessary to have assumption (5.1), which enters into the proof mainly through Claim 2.2 in their paper (which is crucial to their results). In this note, we will prove Claim 2.2 under a weaker assumption than assumption (5.1). More precisely, we prove the following.

Assume

$$\max_{x \in \{-1,0,1\}} \mathbb{P}(\xi_0 = x) < 0.7615\ldots. \tag{5.2}$$

Then there exist constants $C, c > 0$ such that for any $B > 0$ we have

$$\mathbb{P}(P(3) \text{ is divisible by } k^2 \text{ for some } k \geq B) \leq CB^{-c}. \tag{5.3}$$

Remark. The bound in (5.2) is the solution to equation (5.11).

Proof. Fix r such that

$$3^r \leq B^2 < 3^{r+1}. \tag{5.4}$$

Claim.

$$\mathbb{P}(P(3) \text{ is divisible by } k^2 \text{ for some } k \in [B, 2B]) \leq 2^{-cr} \tag{5.5}$$

for some constant $c > 0$.

Proof of Claim. We write

$$P(3) = \sum_{j<r} \xi_j 3^j + \sum_{j=r}^{m} \xi_j 3^j.$$

Fix $\xi_r, \ldots, \xi_m$, and let $\ell = \sum_{j=r}^{m} \xi_j 3^j$.

If k^2 divides $P(3)$, then

$$\sum_{j<r} \xi_j 3^j \equiv -\ell \bmod k^2.$$

Since $\left|\sum_{j<r} \xi_j 3^j\right| < 3^r/2 \leq k^2/2$, we may denote

$$\ell(k) := \sum_{j<r} \xi_j 3^j \in \left(\frac{-k^2}{2}, \frac{k^2}{2}\right)$$

and let

$$S = \big\{\ell(k) : k \in [B, 2B]\big\} \subset \big(-2B^2, 2B^2\big).$$

It follows that

$$\mathbb{P}(P(3) \text{ is divisible by } k^2 \text{ for some } k \in [B, 2B]) \leq \mathbb{P}\Big(\sum_{j<r} \xi_j 3^j \in S\Big). \tag{5.6}$$

Let $\sigma_{(k)} = \big(\sigma_{(k)}(j)\big)_{j=0,\ldots,r-1} \in \{-1, 0, 1\}^r$ be defined by

$$\sum_{j<r} \sigma_{(k)}(j) 3^j = \ell(k)$$

and let

$$A = \{\sigma_{(k)} : k \in [B, 2B]\} \text{ with } |A| \sim \sqrt{3}^{\,r}.$$

Let δ_j be the indicator function of $j, j = -1, 0, 1$, and denote

$$\rho_j := \mathbb{P}(\xi_0 = j) \text{ for } j = -1, 0, 1, \quad \text{and} \quad \rho := \max_j \rho_j.$$

Denote the product measure on $\{-1, 0, 1\}^r$ by

$$\nu := \bigotimes_{j=0}^{r-1} (\rho_0 \delta_0 + \rho_1 \delta_1 + \rho_{-1} \delta_{-1}).$$

Therefore we have the following. (The reasoning is given below the display.)

$$\begin{aligned} \mathbb{P}\big(\sum_{j<r} \xi_j 3^j \in S\big) \leq & \sum_{\sigma \in A} \nu(\sigma) \\ \leq & |A|^{1/p} \bigg(\sum_{\sigma \in A} \nu(\sigma)^q \bigg)^{1/q}, \quad \text{with} \quad \frac{1}{p} + \frac{1}{q} = 1 \\ \lesssim & \sqrt{3}^{\,r/p} \big(\rho_0^q + \rho_1^q + \rho_{-1}^q\big)^{r/q} \\ \leq & \sqrt{3}^{\,r/p} \big(\rho^q + (1-\rho)^q\big)^{r/q}. \end{aligned} \tag{5.7}$$

The second inequality is by Hölder, and the third inequality follows from the following estimate.

$$\begin{aligned} \sum_{\sigma \in A} \nu(\sigma)^q = & \sum_{\sigma \in A} \bigotimes_{j=0}^{r-1} \big(\rho_0 \delta_0(\sigma(j)) + \rho_1 \delta_1(\sigma(j)) + \rho_{-1} \delta_{-1}(\sigma(j))\big)^q \\ = & \sum_{\sigma \in A} \bigotimes_{j=0}^{r-1} \big(\rho_0^q \delta_0(\sigma(j)) + \rho_1^q \delta_1(\sigma(j)) + \rho_{-1}^q \delta_{-1}(\sigma(j))\big) \\ \leq & \sum_{a+b+c=r} \binom{r}{a} \binom{r-a}{b} \rho_0^{aq} \rho_1^{bq} \rho_{-1}^{cq} = (\rho_0^q + \rho_1^q + \rho_{-1}^q)^r. \end{aligned}$$

To finish the proof of the claim, we want to show

$$\sqrt{3}^{\,r/p} \big(\rho^q + (1-\rho)^q\big)^{r/q} < 2^{-cr} \text{ for some constant } c > 0 \tag{5.8}$$

i.e.

$$\sqrt{3}^{\,1/p} \big(\rho^q + (1-\rho)^q\big)^{1/q} < 1,$$

and we want to solve

$$t^q + (1-t)^q = \left(\frac{1}{\sqrt{3}}\right)^{\frac{1}{p-1}}, \quad \text{with} \quad \frac{1}{p} + \frac{1}{q} = 1. \tag{5.9}$$

Let $u = \frac{1}{p-1}$ and rewrite (5.9) as

$$\left(t^{1+u} + (1-t)^{1+u}\right)^{1/u} = \frac{1}{\sqrt{3}} \tag{5.10}$$

Let p go to infinity (hence u goes to 0). Then

$$\begin{aligned} & t^{1+u} + (1-t)^{1+u} \\ &= t(1 + u\log t + O(u^2)) + (1-t)(1 + u\log(1-t) + O(u^2)) \\ &= 1 + \big(t\log t + (1-t)\log(1-t)\big)u + O(u^2). \end{aligned}$$

Hence (5.10) becomes

$$\left(1 + \big(t\log t + (1-t)\log(1-t)\big)u + O(u^2)\right)^{1/u} = \frac{1}{\sqrt{3}}.$$

In the limit for $u \to 0$, we obtain

$$e^{t\log t + (1-t)\log(1-t)} = \frac{1}{\sqrt{3}}.$$

Solving

$$t^t(1-t)^{1-t} = \frac{1}{\sqrt{3}}, \tag{5.11}$$

we obtain $t = 0.7615332817632392\cdots$.

Putting (5.6), (5.7) and (5.8) together gives the claim.

Using the same argument obtaining the second equality in (4.1), we derive (5.3). □

It is possible to exploit somewhat better arithmetical features of the distribution under considerations but gains turn out to be minimal (0.7654 from 0.7615), therefore, will not be elaborated here.

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