(1) $\mathcal{E} = \mathcal{E}_\lambda := \{x \in \mathbb{Z}^2 : |x|^2 = E\}, \quad E = \lambda^2$

Gauss prime + expected number of prime factors

$\Rightarrow$ for generic E, $|\mathcal{E}| =: N \sim \sqrt{\log E} \sim \sqrt{\log \lambda}$ (1.1)

Lemma 1 (Separation property)

For generic E,
$$\min_{X \neq Y \in \mathcal{E}} |X - Y| \gg \frac{\lambda}{(\log \lambda)^{\frac{3}{2}+\varepsilon}} \tag{1.2}$$

pf. Let R be a parameter and $M = R(\log R)^{-\frac{3}{2}-\varepsilon}$. Then

$$\left|\{(X,Y) \in \mathbb{Z}^2 \times \mathbb{Z}^2 : |X| = |Y| \le R \text{ and } |X-Y| < M\}\right| \tag{1.3}$$

$$= \sum_{\substack{V \in \mathbb{Z}^2 \setminus \{0\} \\ |V| < M}} \left|\{X \in \mathbb{Z}^2 : |X| = |X+V| \le R\}\right|$$

$$= \sum_{\substack{V \in \mathbb{Z}^2 \setminus \{0\} \\ |V| < M}} \left|\{X \in B_R : 2X \cdot V + |V|^2 = 0\}\right|$$

$$\lesssim M^2 + \sum_{\substack{V \in \mathbb{Z}^2 \setminus \{0\} \\ |V| < M}} \left|\{|Y| \le 2R : y_1 v_1 + y_2 v_2 = 0\}\right| \tag{1.4}$$

(i) $v_2 = 0 \Rightarrow y_1 = 0$. The contribution to (1.4) is $\sim MR$. Similarly for $v_1 = 0$.

(ii) $v_1 \neq 0 \neq v_2$. Let $d = \gcd(v_1, v_2)$. Hence $(v_1, v_2) = d(v_1', v_2')$, $\gcd(v_1', v_2') = 1$

and $y_1 v_1' + y_2 v_2' = 0$ has $\lesssim \frac{R}{|V'|}$ solutions in Y with $|Y| < R$.

So $(1.4) \lesssim M^2 + 2MR + \sum_{d<R} \sum_{\substack{(v_1, v_2) = d \\ |v| < M}} \frac{R}{|v'|} \lesssim \sum_{d<R} \frac{M}{d} R \lesssim MR \log R$

Hence $(1.3) \lesssim MR \log R$.

On the other hand, $(1.3) \gtrsim \sum'_{\substack{R^2 < E < 2R^2 \\ E \in \mathbb{Z}_+}} 1_{[\min_{|x|^2 = |y|^2 = E,\, x \neq y} |x-y| < M]}$, where

$\sum'_E = \sum_{E = \text{sum of 2 squares}}$, $\quad |\{E \in \mathbb{Z}_+ : R^2 < E < 2R^2,\ E = \text{sum of 2 squares}\}| \sim \frac{R^2}{\sqrt{\log R}}$

$M = R(\log R)^{-\frac{3}{2}-\varepsilon} \Rightarrow$ For generic $E \sim R^2$

$\min_{\substack{|x|^2 = |y|^2 = E \\ x \neq y}} |x-y| \gg R(\log R)^{-\frac{3}{2}-\varepsilon} = \lambda(\log \lambda)^{-\frac{3}{2}-\varepsilon}$

QED.

(2) Given $\gamma : [0,1] \longrightarrow \Gamma \subset \mathbb{T}^2 = \mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z}$ real analytic

$|\dot{\gamma}| \sim 1, \ |\ddot{\gamma}| > c$.

We may assume γ has analytic continuation to ρ-nbd of $[0,1] \subset \mathbb{C}$.

Here $\rho > 0$ is a fixed constant.

Let $\phi(x) = \sum_{\xi \in \mathcal{E}} a_\xi e^{i x \cdot \xi}$, $\quad x \in \mathbb{T}^2$, $\quad \sum |a_\xi|^2 = 1$

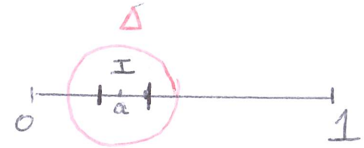
- Want to bound $|Z_\phi \cap \gamma(I)|$, for $I \subset [0,1]$

Here $Z_\phi = \{x \in \mathbb{T}^2 : \phi(x) = 0\}$ $\qquad I \xrightarrow{\gamma} \gamma(I) \subset \Gamma \subset \mathbb{T}^2$

Lemma 2 Let $|I|=\delta<\rho$. Then

$$|Z_\phi \cap \gamma(I)| \lesssim c\delta\lambda + \log N - \log \max_{t\in I}|\phi(\gamma(t))|$$

pf. Let Δ be a disc with the same center a as I and radius 2δ.



For $z\in\Delta$, $\exists\, t\in\mathbb{R}$ s.t. $|z-t|<2\delta$ $\Rightarrow$ $|\gamma(z)-\gamma(t)|\leq c\delta$, when $c=C_\rho$.

Hence for $\xi\in\mathcal{E}$,

$$\begin{aligned} |e^{i\gamma(z)\cdot\xi}| &= |e^{i\xi\cdot(\gamma(z)-\gamma(t))}| \\ &\leq e^{|\xi|\,|\mathrm{Im}(\gamma(z)-\gamma(t))|} \\ &\leq e^{c\lambda\delta} \end{aligned}$$

Therefore

$$|\phi(\gamma(z))| \leq \Big(\sum_{\xi\in\mathcal{E}}|a_\xi|\Big)e^{c\lambda\delta} \underset{\substack{CS \\ \text{and } \sum|a_\xi|^2=1}}{<} \sqrt{N}\, e^{c\lambda\delta}$$

Jenson implies

$$\begin{aligned} |Z_\phi\cap\gamma(I)| \leq |\{z\in\Delta_{a,\delta} : \phi(\gamma(z))=0\}| &\leq \log(\sqrt{N}\,e^{c\lambda\delta}) - \log\max_{t\in I}|\phi(\gamma(t))| \\ &\leq c\lambda\delta + \log N - \log\max_{t\in I}|\phi(\gamma(t))| \end{aligned}$$

QED

- Want to have lower bound on

$$\max_{t\in I}|\phi(\gamma(t))| \quad (3.1) \qquad I\subset[0,1]$$

$$(3.1)^2 \geq \frac{1}{|I|}\int_I |\phi(\gamma(t))|^2\, dt \quad (3.2)$$

Lemma 3 Let $\phi(x) = \sum_{\xi \in \mathcal{E}} a_\xi e^{x \cdot \xi i}$, $\sum |a_\xi|^2 = 1$

Under Separation assumption (Lemma 1), $(3.2) \geq \frac{1}{2}$,

assuming $\delta = |I| > \lambda^{-\frac{1}{2}} (\log \lambda)^{\frac{3}{4}+\varepsilon} N$. (3.3)

$\fint_I = \frac{1}{|I|}\int_I$

Pf $\fint_I |\phi(\gamma(t))|^2 dt = \fint_I \left| \sum a_\xi e^{i\xi \cdot \gamma(t)} \right|^2 dt$

$$= 1 + \sum_{\xi \neq \xi' \in \mathcal{E}} a_\xi \overline{a_{\xi'}} \fint_I e^{i(\xi - \xi')\cdot \gamma(t)} dt$$

$$> 1 - \sum_{\xi \neq \xi'} |a_\xi| |a_{\xi'}| \left| \fint_I e^{i(\xi-\xi')\cdot\gamma(t)} dt \right| \qquad (3.4)$$

- need upper bound on $\fint_I e^{i(\xi-\xi')\cdot\gamma(t)} dt$ for $\xi \neq \xi' \in \mathcal{E}_\lambda$ (3.5)

[5] $\Rightarrow$ $|(3.5)| \lesssim \dfrac{1}{\delta |\xi - \xi'|^{1/2}}$ (3.6)

Hence the 2nd term of (3.4) $\lesssim \dfrac{1}{\delta} \sum_{\xi \neq \xi' \in \mathcal{E}} \dfrac{|a_\xi| |a_{\xi'}|}{|\xi - \xi'|^{1/2}}$

$$\underset{\text{separation assumption}}{\lesssim} \frac{1}{\delta \lambda^{1/2}} (\log \lambda)^{\frac{3}{4}+\varepsilon} N \underset{\text{by (3.3)}}{\lesssim} 1$$

Hence $(3.4) \geq \frac{1}{2}$. QED

Consider $I = [a, a+\delta]$, $\delta \leq \lambda^{-1+\varepsilon}$. Write

$$(\xi-\xi')\cdot\gamma(t) = (\xi-\xi')\cdot\gamma(a) + (\xi-\xi')\cdot\gamma'(a)\tau + O(|\xi-\xi'|\delta^2),$$

when $t = a+\tau$ and $|\xi-\xi'|\delta^2 \ll \lambda\lambda^{-2+2\varepsilon} \ll \lambda^{-1+\varepsilon}$.

$$\left|\int_0^\delta e^{i((\xi-\xi')\cdot\dot\gamma(a))\tau}\,d\tau\right| \lesssim \frac{1}{|(\xi-\xi')\cdot\dot\gamma(a)|} \quad (3.7)$$

Assume $|(\xi-\xi')\cdot\dot\gamma(a)| >> |\xi-\xi'|\delta$ (3.8)

Then $$\left|\fint_I e^{i(\xi-\xi')\cdot\gamma(t)}\,dt\right| \lesssim \frac{1}{\delta\,|(\xi-\xi')\cdot\dot\gamma(a)|} \quad (3.9)$$

Consider the set of directions

$$\mathcal{D} = \left\{\frac{\xi-\xi'}{|\xi-\xi'|} : \xi \neq \xi' \in \mathcal{E}\right\} \qquad |\mathcal{D}| < N^2$$

Let κ be a parameter $\frac{c}{N^2} > \kappa > \lambda^{-\varepsilon}$.

We partition $[0,1]$ (hence P) as follows.

$\forall$ unit direction φ

$\{t \in [0,1] : \text{angle}(\dot\gamma(t), \varphi) < \kappa\}$ ie an interval of size $\lesssim \kappa$.

Hence

$$[0,1] = \bigcup_{\alpha < N^2} I_\alpha \cup \bigcup_{\beta < N^2} J_\beta,$$

where I_α, J_β are intervals s.t. $|I_\alpha| \lesssim \kappa \quad \forall\alpha$

$|J_\beta| \gtrsim \kappa \quad \forall\beta$

$\forall \beta$ and $\forall t \in J_\beta$, $\text{angle}(\dot{\gamma}(t), \xi) > \kappa$ $\forall \xi \in \mathcal{D}^\perp$.

- We expand I_α to an interval $\tilde{I}_\alpha$ of size κ.

(Note that $\kappa > \lambda^{-\varepsilon}$, hence (3.3) holds.) Lemma 3 implies

$$\max_{t \in \tilde{I}_\alpha} |\phi(\gamma(t))|^2 \geq \int_{\tilde{I}_\alpha} |\phi(\gamma(t))|^2 \geq \frac{1}{2}.$$

Lemma 2 implies

$$|Z_\phi \cap \gamma(I_\alpha)| \leq |Z_\phi \cap \gamma(\tilde{I}_\alpha)| \lesssim \kappa\lambda + \log N - c \lesssim \kappa\lambda$$

Therefore,

$$\sum_\alpha |Z_\phi \cap \gamma(I_\alpha)| \lesssim N^2 \kappa \lambda.$$

Take $\kappa = N^{-2-c}$. Hence

$$|Z_\phi \cap \gamma(\bigcup_\alpha I_\alpha)| < \lambda N^{-c} \quad \text{(an error term)} \tag{3.10}$$

- Now consider any $I \subset [0,1]$, $|I| = \frac{1}{\lambda}$, $I \cap \bigcup_\alpha I_\alpha = \emptyset$

wma $I = [a, a + \frac{1}{\lambda}]$ $a \in J_\beta$ for some β.

Denote $\tilde{I} = [a, a + \frac{M}{\lambda}]$, where $M = N^7$ ($\Rightarrow$ $M > (\log \lambda)^{\frac{3}{2}+\varepsilon} N^{3+c}$)

Lemma 2 implies

$$|Z_\phi \cap \gamma(I)| \leq |Z_\phi \cap \gamma(\tilde{I})| \lesssim c\frac{M}{\lambda}\lambda + \log N - \log \max_{t \in \tilde{I}} |\phi(\gamma(t))| \tag{3.11}$$

Since $a \in J_\beta$, angle $(\dot{\gamma}(a), \zeta) > \kappa$ $\forall \zeta \in \mathcal{D}^\perp$ so

$\forall \xi \neq \xi' \in \mathcal{E}$, $|(\xi-\xi')\cdot\dot{\gamma}(a)| > \kappa|\xi-\xi'| >> \delta|\xi-\xi'|$.

Thus (3.8) holds, and
$$\left| \int_{\tilde{I}} e^{i(\xi-\xi')\cdot\gamma(t)}\,dt \right| \underset{(3.9)}{\lesssim} \frac{1}{\delta\cdot\kappa|\xi-\xi'|}$$

$$= \frac{\lambda}{M\,\kappa|\xi-\xi'|}$$

$\left(\text{since } |\hat{I}| = \delta = \frac{M}{\lambda}\right)$

Lemma 1 $\Rightarrow |\xi-\xi'| >> \dfrac{\lambda}{(\log\lambda)^{\frac{3}{2}+\varepsilon}}$ for $\xi \neq \xi' \in \mathcal{E}$

Hence
$$\left| \int_{\tilde{I}} e^{i(\xi-\xi')\cdot\gamma(t)}\,dt \right| \lesssim \frac{(\log\lambda)^{\frac{3}{2}+\varepsilon}\, N^{2+c}}{M} \qquad (3.12)$$
(by choice of κ on P.5)

(3.4) & (3.12) give

$$\int_{\tilde{I}} |\phi(\gamma(t))|^2\,dt > 1 - c\,\frac{(\log\lambda)^{\frac{3}{2}+\varepsilon}\, N^{2+c}}{M}\cdot\left(\sum|a_\xi|\right)^2$$

$$> \frac{(\log\lambda)^{\frac{3}{2}+\varepsilon}\, N^{\beta+c}}{M} > \frac{1}{2}$$
P.6 $M = N^7$ and $N \sim \sqrt{\log\lambda}$

$$(3.2) \Longrightarrow \max_{t\in\tilde{I}} |\phi(\gamma(t))| > \frac{1}{\sqrt{2}} \qquad (3.13)$$

$$(3.11),\ (3.13) \Longrightarrow |Z_\phi \cap \gamma(I)| \lesssim M + \log N - c \lesssim N^7 \qquad (3.14)$$

This bound (3.14) (for generic E) implies "condition (4)"

for interval $I \subset [0,1] \setminus \bigcup_\alpha I_\alpha$, $|I| \sim \frac{1}{T}$

From (3.10), the contribution from $\bigcup_\alpha I_\alpha$ is negligible.