ON PRODUCT SETS IN FIELDS OF PRIME ORDER AND AN APPLICATION OF BURGESS' INEQUALITY

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This is the origin of paper 'On a Question of Davenport and Lewis on Character Sums and Primitive Roots in Finite Fields'. There is still a little to be typed.

Abstract Let $A \subset \mathbb{F}_p$ with $|A| > p^{2/5+\varepsilon}$ and $|A+A| < C_0|A|$. We give explicit constants $k = k(C_0, \varepsilon)$ and $\kappa = \kappa(C_0, \varepsilon)$ such that $|A^k| > \kappa p$. The tools we use are Garaev's sum-product estimate, Freiman's Theorem and a variant of Burgess' method. As a by-product, we also obtain similar result for proper generalized progression in $\mathbb{F}_p$.

§0. Introduction.

Let A be a subset of $\mathbb{F}_p$. The *sum set* and the *k-fold product set* of A are

$$A + A = \{a + b \,:\, a, b \in A\}$$

and

$$A^k = A \cdots A = \{a_1 \cdots a_k \,:\, a_1, \cdots, a_k \in A\},$$

respectively. We give explicit bounds on k and κ such that for a large set A, the size of the k-fold product set is at least κp. The bounds depend on the size and the doubling constant of A. More precisely, we prove the following theorem.

Theorem 1. *Let A be a subset of $\mathbb{F}_p$ satisfying the following properties*

(i) $|A + A| < C_0|A|$

(ii) $|A| > p^{2/5+\varepsilon}$.

Then there are constants $k = k(C_0, \varepsilon) \in \mathbb{Z}^+$ and $\kappa = \kappa(C_0, \varepsilon)$ such that

$$|A^k| > \kappa p.$$

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Explicit bounds on k and κ are in Remarks 5 and 6.

It is possible that the conclusion of Theorem 1 holds under assumption (i) and the much weaker hypothesis that $|A| > p^\varepsilon$, for any given $\varepsilon > 0$. But even if A is an interval $I \subset [1,p)$, with $|I| > p^\varepsilon$, it seems unknown whether $|I^k| > cp$ for appropriate $k = k(\varepsilon)$, unless I is of the form $[1,a]$ with $a > p^\varepsilon$. (One may then proceed by simple arithmetical considerations.) We would also like to point out that by Burgess' estimate, assuming $|I| > p^{\frac{1}{4}+\varepsilon}$, one gets $I^k \supset \mathbb{F}_p^*$ for $k > k(\varepsilon)$. If moreover, I is of the form $[1,a]$, the condition that $a > p^{\frac{1}{4\sqrt{e}}+\varepsilon}$ suffices. It is not obvious to what extent that Burgess's method may be generalized to sets other than intervals. In this paper, we consider the case of sets A with small doubling $|A+A| < c|A|$. This situation was also considered in [HIS] and we significantly improved their result by involving several additional ingredients. Under the assumption that $|A+A| < c|A|$ Freiman's theorem is applicable and essentially reduces the issue to higher dimensional arithmetic progressions. However, in order to carry out Burgess' amplification argument, we also reply on recent quantitative versions of the general sum-product theorem in $\mathbb{F}_p$. (See [G], [KS].) In this context, let us bring up the following question

Question. Fix d and let $\mathcal{P}$ be a generalized d-dimensional arithmetic progression in $\mathbb{F}_p$. Assume that $|\mathcal{P}| < \sqrt{p}$. Is it true that $|\mathcal{P}\mathcal{P}| \gg |\mathcal{P}|^{2-\varepsilon}$ for all $\varepsilon > 0$?

From the result in [KS], we know that $|\mathcal{P}\mathcal{P}| \gg |\mathcal{P}|^{\frac{5}{4}-\varepsilon}$ as well as the corresponding statemnet for multiplicative energy. This allows us to obtain the following variant of Burgess' inequality.

Theorem 2. *Fix d and let $\mathcal{P}$ be a proper generalized d-dimensional progression in $\mathbb{F}_p$ with $|\mathcal{P}| > p^{\frac{2}{5}+\varepsilon}$. Then for some $\tau = \tau(\varepsilon, d)$*

$$\max_\psi \Big| \sum_{x\in\mathcal{P}} \psi(x) \Big| < p^{-\tau}|\mathcal{P}|,$$

where ψ runs over nontrivial multiplicative characters.

Note We use A^n for both the n-fold product set and n-fold Cartesian product when there is no ambiguity.

§1. A consequence of Garaev's sum-product estimate.

We will follow the argument in [KS], where the authors improved Garaev's estimate of a lower bound on $|A+A| + |AA|$ from $\frac{15}{14}$ to $\frac{14}{13}$ for $A \subset \mathbb{F}_p$ with $|A| < p^{1/2}$. Specifically, we will use the following estimate. $\big($See (2.5)$-\varepsilon$ in [KS].$\big)$

Proposition KS. [KS] *Let* $A \subset \mathbb{F}_p$ *with* $|A| < p^{\frac{1}{2}}$. *If there exist* $b_0 \in A$, $A_1 \subset A$ *and* $N \in \mathbb{Z}^+$ *satisfying*

$$|aA \cap b_0 A| \sim N \text{ for every } a \in A_1, \tag{1.1}$$

then

$$|A_1|^3 N^4 |A|^3 \lesssim |A+A|^9. \tag{1.2}$$

Let the multiplicative energy of A be

$$E(A) = |\{(a_1.a_2, a_3, a_4) \in A^4 : a_1 a_2 = a_3 a_4\}|.$$

(See [TV].) The following is a consequence of Proposition KS.

Lemma 3. *Let* $A \subset \mathbb{F}_p$ *with* $|A| < p^{\frac{1}{2}}$. *Then*

$$c|A+A|\ |A|^2 > E(A)^4.$$

Proof. We write $E(A)$ as

$$E(A) = \sum_{a,b \in A} |aA \cap bA|.$$

Take $b_0 \in A$ with

$$\sum_{a \in A} |aA \cap b_0 A| \geq \frac{E(A)}{|A|}. \tag{1.3}$$

We decompose A into level sets

$$A_s = \{a \in A : 2^{s-1} \leq |aA \cap b_0 A| < 2^s\},$$

where $0 \leq s \leq \log_2 |A|$.

Thus by (1.3)

$$\sum_s 2^s\ |A_s| \geq \frac{E(A)}{|A|}. \tag{1.4}$$

For $j \in \mathbb{Z}^+$, let

$$s_j = \max\{s : s^{j-1} \leq |A_s| < 2^j\}.$$

Clearly,

$$\sum_s 2^s\ |A_s| \sim \sum_j 2^{s_j} 2^j. \tag{1.5}$$

Also, we have

$$\sum_j 2^{s_j} 2^j \le \big(\max_j 2^{s_j} 2^{\frac{3}{4}j}\big) \sum_{j=0}^{\log_2 |A|} 2^{\frac{j}{4}} \le \max_s 2^s \ |A_s|^{\frac{3}{4}} \ |A|^{\frac{1}{4}}. \tag{1.6}$$

Combining (1.4), (1.5) and (1.6), we have

$$\max_s 16^s \ |A_s|^3 \gtrsim \frac{E(A)^4}{|A|^5}. \tag{1.7}$$

Now for $s = 0, \cdots, \log_2 |A|$, applying Proposition KS with $N = 2^s$ and $A_1 = A_s$, we have

$$|A_s|^3 16^s \ |A|^3 \lesssim |A+A|^9. \tag{1.8}$$

Combined with (1.7), this gives

$$\frac{E(A)^4}{|A|^2} \lesssim |A+A|^9. \qquad \square$$

§2. The proof of Theorem 1.

By Freiman's Theorem [C], under assumption (i), there is a proper generalized d-dimensional progression $\mathcal{P}$ such that $A \subset \mathcal{P}$ and

$$d < C_0 \tag{2.1}$$

$$\log \frac{|\mathcal{P}|}{|A|} < C_1 = C_0^2 (\log C_0)^3. \tag{2.2}$$

Write

$$\mathcal{P} = \xi + I_{\ell_1}\xi_1 + \cdots + I_{\ell_d}\xi_d,$$

where $\xi, \xi_1, \cdots, \xi_d \in \mathbb{F}_p$, and $I_{\ell_j} = \{0, 1, \cdots, \ell_j - 1\}$ with $\ell_1 \ge \ell_2 \cdots \ge \ell_d$ and

$$\ell_1 \ell_2 \cdots \ell_d = |\mathcal{P}|. \tag{2.3}$$

Let

$$\delta = \frac{\varepsilon}{d}, \tag{2.4}$$

and let $d \ge d' \ge 1$ characterized by the property that

$$\ell_i \ge p^\delta, \text{ if } i \le d'$$

$$\ell_i < p^\delta, \text{ if } i > d'. \tag{2.5}$$

Denote

$$m = \big[p^{\frac{\delta}{2}}\big],$$

$$m_i = \big[p^{-\frac{\delta}{2}}\ell_i\big], \text{ for } i \le d',$$

and

$$\mathcal{Q} = I_{m_1}\xi_1 + \cdots + I_{m_{d'}}\xi_{d'}. \tag{2.6}$$

Hence by (2.5),

$$|\mathcal{Q}| \ge p^{-\frac{\delta}{2}d'} \prod^{d'} \ell_i > p^{-\frac{\delta}{2}d' - \delta(d-d')}|\mathcal{P}| > p^{-d\delta}|\mathcal{P}| \tag{2.7}$$

and

$$I_m\ \mathcal{Q} \subset I_{\ell_1}\xi_1 + \cdots + I_{\ell_{d'}}\xi_{d'} \subset \mathcal{P} - \xi. \tag{2.8}$$

Consider the map

$$\begin{aligned} \phi : I_m \times \mathcal{Q} \times (\mathcal{P} - \mathcal{P} + \xi) &\longrightarrow \mathbb{F}_p \\ (z, x, y) &\mapsto y + zx. \end{aligned}$$

We denote by ν the image measure on $\mathbb{F}_p$ of the normalized counting measure on $I_m \times \mathcal{Q} \times (\mathcal{P} - \mathcal{P} + \xi)$ under ϕ. Thus for $t \in \mathbb{F}_p$

$$\nu(t) = \nu(\{t\}) = \frac{1}{|I_m|\ |\mathcal{Q}|\ |\mathcal{P}-\mathcal{P}|}\ \big|\{(z,x,y) \in I_m \times \mathcal{Q} \times (\mathcal{P}-\mathcal{P}+\xi) : t = y + zx\}\big|. \tag{2.9}$$

Define

$$\nu_1(t) = \nu(t) \text{ for } t \in \mathbb{F}_p^*, \text{ and } \nu_1(0) = 0. \tag{2.10}$$

We may assume $A \subset \mathbb{F}_p^*$.

Similarly, we define $\nu_2(t) = \nu(t)$ for $t \in A$, and $\nu_2(t) = 0$ for $t \in \mathbb{F}_p \setminus A$.

Since for $t \in A$ and any $z \in I_m, x \in \mathcal{Q}$, we have $t - zx \in A - I_m\ \mathcal{Q} \subset \mathcal{P} - \mathcal{P} + \xi$ by (2.8). It follows that

$$\nu(t) = \nu_1(t) \ge \frac{1}{|\mathcal{P} - \mathcal{P}|} \text{ for } t \in A.$$

Therefore, by (2.2)

$$\nu(A) = \nu_1(A) \ge \frac{|A|}{|\mathcal{P} - \mathcal{P}|} > e^{-C_1 - d}. \tag{2.11}$$

We denote the multiplicative convolution of two functions f and g on $\mathbb{F}_p^*$ by

$$(f \otimes g)(x) = \sum_{y \in \mathbb{F}_p^*} f(xy^{-1}) g(y).$$

Also, we denote further the k-fold multiplicative convolution of ν_i by $\nu_i^{(k)}$ for $i = 1, 2$. In the next section we will prove the following proposition.

Proposition 4. *Let $A \subset \mathbb{F}_p$ with $|A| > p^{\frac{2}{5}+\varepsilon}$ and let ν_1 be defined as in (2.1)-(2.10). Then there exists $k = k(\varepsilon, C_0)$ such that*

$$\nu_1^{(k)}(x) \le \frac{2}{p}$$

for all $x \in \mathbb{F}_p^*$.

Remark 5. As indicated in (3.16), we can take $k = \frac{16C_0}{\varepsilon^2}$.

Proposition 4 implies

$$\nu_2^{(k)}(x) \le \frac{2}{p}.$$

Hence

$$\nu_2^{(k)}(x) \le \frac{2}{p}\, \chi_{_{A^k}}. \tag{2.12}$$

Therefore, by (2.12)

$$\frac{2}{p}\, |A^k| \ge \sum_{x \in \mathbb{F}_p^*} \nu_2^{(k)}(x) = \big(\sum_{x \in \mathbb{F}_p^*} \nu_2(x) \big)^k = \nu(A)^k > e^{-(C_1+d)k}$$

and hence

$$|A^k| > e^{-(C_1+C_0)k} p, \tag{2.12}$$

which proves the Theorem.

Remark 6. Combining (2.2), (2.12) and Remark 5, we have an explicit bound on κ in our theorem

$$\kappa > e^{-16C_0^2 \big(C_0 (\log C_0)^3 + 1 \big)/\varepsilon^2}.$$

§3 The proof of Proposition 4.

Let ψ be a multiplicative character mod p and let f be a function on $\mathbb{F}_p^*$. Then

$$f(x) = \frac{1}{p-1}\sum_{\psi} \hat{f}(\psi)\psi(x),$$

where

$$\hat{f}(\psi) = \sum_{x\in\mathbb{F}_p^*} f(x)\overline{\psi(x)}.$$

Also

$$(f\otimes g)\hat{\ }(\psi) = \hat{f}(\psi)\hat{g}(\psi).$$

Hence

$$\begin{aligned}\nu_1^{(k)}(x) &= \frac{1}{p-1}\ \sum_{\psi} \widehat{\nu_1}(\psi)^k\psi(x)\\ &= \frac{1}{p-1}\ \nu_1(\mathbb{F}_p^*)^k + \frac{1}{p-1}\ \sum_{\psi \text{ nontrivial}} \widehat{\nu_1}(\psi)^k\psi(x).\end{aligned}$$

Therefore,

$$\nu_1^{(k)}(x) < \frac{1}{p-1} + \max_{\psi \text{ nontrivial}} |\widehat{\nu_1}(\psi)|^k.$$

To prove Proposition 4, it suffices to show that for some k

$$\max_{\psi \text{ nontrivial}} |\widehat{\nu_1}(\psi)|^k < \frac{1}{p}\ .$$

This will follow from that

$$\max_{\psi \text{ nontrivial}} |\widehat{\nu_1}(\psi)| < p^{-\tau} \tag{3.1}$$

for some $\tau = \tau(\varepsilon, C_0) > 0$.

By (2.9), estimate (3.1) is equivalent to

$$\Big|\sum_{\substack{z\in I_m, x\in\mathcal{Q}\\ y\in\mathcal{P}-\mathcal{P}+\xi}} \psi(y+zx)\Big| < p^{-\tau}|I_m|\ |\mathcal{Q}|\ |\mathcal{P}-\mathcal{P}|. \tag{3.2}$$

The following is a version of Proposition 4 with slightly general setting.

Proposition 4'. *Let* $I_m = \{1, \cdots, m\}$*,* ψ *be a multiplicative character and* $\mathcal{Q}$ *and* $\mathcal{R}$ *be two progressions in* $\mathbb{F}_p$ *with length bounded by* $p^{\frac{1}{2}}$ *and dimension bounded by* d*. Then for any* $q \in \mathbb{Z}^+$*,*

$$\Big|\sum_{\substack{z\in I_m, x\in\mathcal{Q}\\ y\in\mathcal{R}}} \psi(y+zx)\Big| < c^{\frac{d}{q}}(|\mathcal{Q}|\,|\mathcal{R}|)^{1-\frac{5}{16q}}(q\sqrt{m}+mp^{\frac{1}{4q}}).$$

Proof. We will adapt the method of Burgess' estimate. By the Hölder's inequality,

$$\Big|\sum_{\substack{z\in I_m, x\in\mathcal{Q}\\ y\in\mathcal{R}}} \psi(y+zx)\Big| \le \sum_{\substack{x\in\mathcal{Q}\\ y\in\mathcal{R}}}\Big|\sum_{z\in I_m}\psi(y+zx)\Big|$$
$$\le |\mathcal{Q}|^{1-\frac{1}{q}}|\mathcal{R}|^{1-\frac{1}{q}}\Big(\sum_{\substack{x\in\mathcal{Q}\\ y\in\mathcal{R}}}\Big|\sum_{z\in I_m}\psi(\frac{y}{x}+z)\Big|^q\Big)^{\frac{1}{q}}. \qquad (3.3)$$

We denote by η the image measure on $\mathbb{F}_p$ of the normalized counting measure on $\mathcal{Q}\times\mathcal{R}$ under the map $(x, y) \mapsto \frac{y}{x}$,

$$\eta(t) = \eta(\{t\}) = \frac{1}{|\mathcal{Q}|\,|\mathcal{R}|}\,\big|\{(x,y)\in\mathcal{Q}\times\mathcal{R} : t = \frac{y}{x}\}\big|.$$

Thus, by Cauchy-Schwarz inequality the bound in (3.3) is

$$|\mathcal{Q}|\,|\mathcal{R}|\left(\sum_{t\in\mathbb{F}_p}\eta(t)\,\Big|\sum_{z\in I_m}\psi(t+z)\Big|^q\right)^{\frac{1}{q}}$$
$$\le |\mathcal{Q}|\,|\mathcal{R}|\,\Big(\sum_{t\in\mathbb{F}_p}\eta(t)^2\Big)^{\frac{1}{2q}}\left(\sum_{t\in\mathbb{F}_p}\Big|\sum_{z\in I_m}\psi(t+z)\Big|^{2q}\right)^{\frac{1}{2q}}. \qquad (3.4)$$

To obtain an upper bound on $\sum\eta(t)^2$ in (3.4), we will use Cauchy-Schwarz inequality and express the upper bound by the multiplicative energies of $\mathcal{Q}$ and $\mathcal{R}$.

$$
\begin{aligned}
&|\mathcal{Q}|^2|\mathcal{R}|^2\sum\eta(t)^2\\
&=\left|\left\{(x_1,x_2,y_1,y_2)\in\mathcal{Q}^2\times\mathcal{R}^2:\frac{y_1}{x_1}=\frac{y_2}{x_2}\right\}\right|\\
&=\left|\left\{(x_1,x_2,y_1,y_2)\in\mathcal{Q}^2\times\mathcal{R}^2:\frac{y_1}{y_2}=\frac{x_1}{x_2}\right\}\right|\\
&=\sum_t\left|\left\{(x_1,x_2)\in\mathcal{Q}^2:\frac{x_1}{x_2}=t\right\}\right|\left|\left\{(y_1,y_2)\in\mathcal{R}^2:\frac{y_1}{y_2}=t\right\}\right|\\
&\le\left(\sum_t\left|\left\{(x_1,x_2)\in\mathcal{Q}^2:\frac{x_1}{x_2}=t\right\}\right|^2\right)^{\frac12}\left(\sum_t\left|\left\{(y_1,y_2)\in\mathcal{R}^2:\frac{y_1}{y_2}=t\right\}\right|^2\right)^{\frac12}\\
&=E(\mathcal{Q})^{\frac12}E(\mathcal{R})^{\frac12}. \qquad (3.5)
\end{aligned}
$$

We apply Lemma 3 to $\mathcal{Q}$ and $\mathcal{R}$. Since $\mathcal{Q},\mathcal{R}$ are progressions,

$$E(\mathcal{Q})^4<c|\mathcal{Q}+\mathcal{Q}|^9|\mathcal{Q}|^2<c^d|\mathcal{Q}|^{11}, \qquad (3.6)$$

$$E(\mathcal{R})^4<c^d|\mathcal{R}|^{11}. \qquad (3.7)$$

Substituting (3.6) and (3.7) in (3.5) gives

$$\sum\eta(t)^2<c^d|\mathcal{Q}|^{-\frac58}|\mathcal{R}|^{-\frac58}. \qquad (3.8)$$

Now we estimate the character sum in (3.4). First, observe that

$$\sum_{t\in\mathbb{F}_p}\Big|\sum_{z\in I_m}\psi(t+z)\Big|^{2q}=\sum_{z_1,\cdots,z_{2q}\in I_m}\ \sum_{t\in\mathbb{F}_p}\psi\left(\frac{(t+z_1)\cdots(t+z_q)}{(t+z_{q+1})\cdots(t+z_{2q})}\right). \qquad (3.9)$$

By Weil's estimate [W] (see also Theorem 11.24 in [IK])

$$\left|\sum_{t\in\mathbb{F}_p}\psi\left(\frac{(t+z_1)\cdots(t+z_q)}{(t+z_{q+1})\cdots(t+z_{2q})}\right)\right|<2q\sqrt{p}\,, \qquad (3.10)$$

unless there exists $1\le i\le 2q$ such that $z_i\ne z_j$ for all $j\ne i$.

Therefore

$$\Big|\sum_{t\in\mathbb{F}_p}\Big|\sum_{z\in I_m}\psi(t+z)\Big|^{2q}\le q^{2q}m^q+2qm^{2q}\sqrt{p}. \qquad (3.11)$$

The proposition is proved by putting (3.3), (3.4), (3.8) and (3.11) together. □

Returning to the proof of Proposition 4.

Assuming that $|A| = p^{2/5+\varepsilon}$ (which we may), it follows in particular, from (2.2) and (2.6) that $|\mathcal{P}-\mathcal{P}| \sim |\mathcal{P}| \sim |A| < p^{1/2}$ and $|\mathcal{Q}| < p^{1/2}$

We apply Proposition to the case of which $\mathcal{R} = \mathcal{P}-\mathcal{P}+\xi$. By (2.7), we have

$$\begin{aligned}
&\Big|\sum_{\substack{z\in I_m, x\in\mathcal{Q}\\ y\in\mathcal{P}-\mathcal{P}+\xi}} \psi(y+zx)\Big| \\
&< c^d|\mathcal{Q}|\,|\mathcal{P}|\,|\mathcal{P}|^{-\frac{5}{16q}}|\mathcal{Q}|^{-\frac{5}{16q}}(q\sqrt{m}+mp^{\frac{1}{4q}}) \\
&< c^d|\mathcal{P}|\,|\mathcal{Q}|\,m\Big(q\,\frac{1}{\sqrt{m}}\Big(\frac{p^{\frac{5}{8}d\delta}}{|\mathcal{P}|^{\frac{5}{4}}}\Big)^{\frac{1}{2q}} + \Big(\frac{p^{1+\frac{5}{4}d\delta}}{|\mathcal{P}|^{\frac{5}{2}}}\Big)^{\frac{1}{4q}}\Big) \\
&< c^d|\mathcal{P}|\,|\mathcal{Q}|\,m\Big(\sqrt{q}\,p^{\frac{1}{2q}(-\frac{1}{2}-\frac{5}{4}\varepsilon+\frac{5}{8}d\delta)-\frac{\delta}{4}} + p^{-\frac{5}{8q}(\varepsilon-\frac{d\delta}{2})}\Big).
\end{aligned} \tag{3.12}$$

The last inequality is because $|\mathcal{P}| \geq |A| = p^{2/5+\varepsilon}$ and $m = \big[p^{\frac{\delta}{2}}\big]$.

Finally, we choose our parameters.

Take

$$\delta = \frac{\varepsilon}{d} \tag{3.13}$$

$$q = \Big[\frac{5d}{\varepsilon}\Big]. \tag{3.14}$$

Then (3.12) implies (3.2) with

$$\tau = \frac{\varepsilon^2}{16d} > \frac{\varepsilon^2}{16C_0}. \tag{3.15}$$

This completes the proof of the Theorem. Moreover, since in (3.1), $k \sim \frac{1}{\tau}$, we have

$$k < \frac{16C_0}{\varepsilon^2}. \tag{3.16}$$

References

[C]. M-C. Chang, *A polynomial bound in Freiman's Theorem,*, Duke Math. J. 113, No 3, (2002), 399-419.

[G]. M. Garaev, *An explicit sum-product estimate in* $\mathbb{F}_p$, (preprint).

[HIS]. D. Hart, A. Iosevich, J. Solymosi, *Sum-product estimates in finite fields via Kloosterman sums*, International Mathematics Research Notices(to appear).

[IK]. H. Iwaniec, E. Kowalski, *Analytic Number Theory*, AMS Colloquium Publications, Volume 53, 2004.

[KS]. N.H Katz, C-Y. Shen, *A slight improvement of Garaev's sum product estimate, preprint.*

[TV]. T. Tao, V. Vu, *Additive Combinatorics*, Cambridge University Press, 2006.

[W]. A. Weil, *On some exponential sums*, Proc. Nat. Acad. Sci. U.S.A. 34 (1948) 204-207.

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