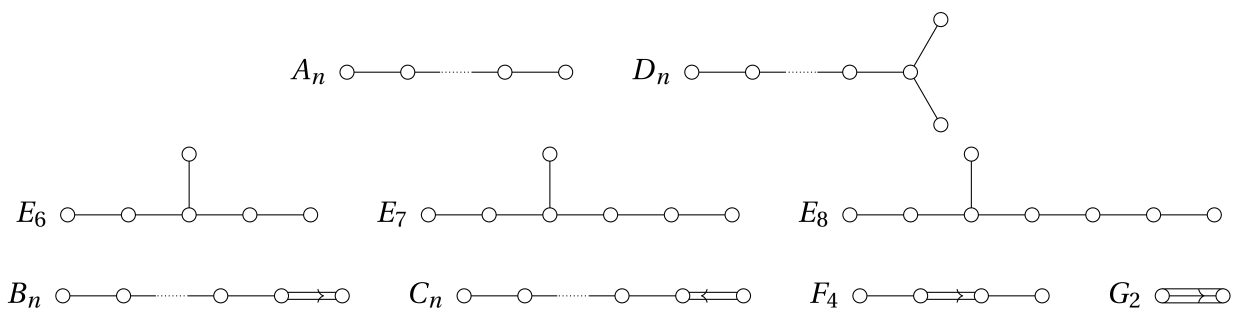
Research Statement

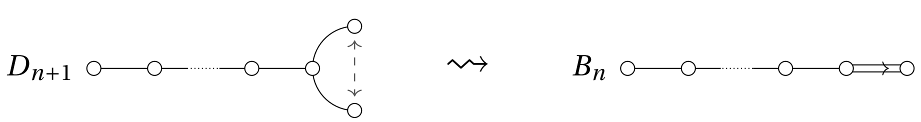
April 28, 2020

I work in the broad field of mathematics called *representation theory*, where we study intricate mathematics objects, ones that model physical phenomenon or ones that have sprung from purely mathematical work, by looking at their symmetries. My particular research involves describing how the "folding" of a Lie algebra induces a similar folding of a quantum group affiliated with that Lie algebra in terms of the representation theory of quivers. To unpack that, first we need to know that the simple complex Lie algebras are completely classified, and each correspond to one of these diagrams and a choice of n for the number of nodes.



These are the *Dynkin diagrams*. Notice that in the bottom row, the diagrams have a double- or triple-edge. The top two rows, the diagrams of type A, D, and E are called the *simply-laced Dynkin diagrams* and are ubiquitous in their own right. They classify an extensive list of different objects in mathematics and physics, a few notable examples being connected quivers of finite representation type, symmetry groups associated to regular platonic solids, and two-dimensional conformal field theories.

Each of the diagrams in the bottom row can be drawn by "folding" one of the simply-laced diagrams along a line of symmetry. For example there is a symmetry in the family of diagrams D_{n+1} between the two "fork" nodes of the diagram: you can fold a D_{n+1} diagram by pushing together those two fork nodes into a single node, creating a diagram of type B_n.



Now this folding not only makes sense looking at the diagrams, but is also naturally meaningful in terms of the Lie algebras that they represent: a Lie algebra $\mathfrak{so}_{2n+2}$ corresponding to the D_{n+1} diagram has a symmetry just like the diagram does, and the elements of the algebra that are fixed under that symmetry form the Lie algebra $\mathfrak{so}_{2n+1}$ corresponding to B_n. That is, this folding $D_n \rightsquigarrow B_{n-1}$ gives us an inclusion of algebras $\mathfrak{so}_{2n+1} \hookrightarrow \mathfrak{so}_{2n+2}$. The BIG QUESTION then is, for all the objects like the ones listed above that just the simply-laced diagrams classify, how can we interpret this folding operation? My research assists in answering this broad question.

While the implications of this diagram folding is understood in many situations, it's not quite flushed out in the case of the *quantized enveloping algebras* $\mathbf{U}_q(\mathfrak{g})$ that each correspond to a Lie algebra $\mathfrak{g}$. These algebras $\mathbf{U}_q(\mathfrak{g})$ are the quintessential examples of quantum groups[2], objects that have become the center of study to many mathematicians—representation theorists, quantum algebraists, and mathematics physicists in particular. To follow our example, we want to know what kind of relationship the diagram folding builds between $\mathbf{U}_q(\mathfrak{so}_{2n+2})$ and $\mathbf{U}_q(\mathfrak{so}_{2n+1})$.

Berenstein and Greenstein[1] describe what this folding of Lie algebras means in terms of their quantized enveloping algebras. While the authors prove that an analogue of the folding exists and describe some rich algebraic structure associated to the induced inclusion of quantum groups, they don't have a clean description of these folded algebras themselves. Instead they only write out a few examples explicitly in the paper, manually computing presentations of the algebras case-by-case. My research aims to describe the structures of these folded algebras more elegantly, by using a result of Ringel[5][6] that a quantized enveloping algebra $\mathbf{U}_q(\mathfrak{g})$ has basically the same structure as the *Hall algebra*[4][3] built from the representation theory of a quiver that corresponds to the Lie algebra $\mathfrak{g}$. Being able to describe the quantum folding of an algebra $\mathbf{U}_q(\mathfrak{g})$ simply as the Hall algebra of a specific quiver is certainly more helpful than working with it in terms of an unwieldy list of generating elements and relations.

And this is helpful not only to the mathematicians and physicists who work with these algebras in their research, but also helpful to math and physics students coming after me. The whole point of my research is to describe certain $\mathbf{U}_q(\mathfrak{g})$ more simply as "foldings" of other ones, which has clear pedagogical benefits to future students trying to understand these objects themselves.

Now while mathematicians and physicists have an interest in this research, it's tough to describe an immediate benefit it will have for people outside these fields. But this is a feature of nearly all pure mathematics research. Historically the benefits of many mathematical ideas to society, for examples the application of number theory to cryptography, or the recent idea to use homology to pick out trends in high dimensional sets of data (see *topological data analysis*), have only come long after their conception, and were unforeseeable to the mathematicians that conceived them. And so it's impossible to say how my research will contribute to society in the future, especially since it may very well be beyond my lifetime that it finds a concrete application.

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