Algebra – Set Theory & Functions

Advanced Mathematics Program, Summer 2019

Using Venn diagrams, convince yourself that each of the following statements is true and then prove them using the methods discussed in lecture.

- $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.
- $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

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1. Suppose $A \subseteq B$ and $B \subseteq C$. Prove that $A \subseteq C$.
2. (a) Prove that $A \cap B \subseteq A$ and $A \cap B \subseteq B$.

   (b) Prove that $A \subseteq A \cup B$ and $B \subseteq A \cup B$.
3. Prove that $(A \cap B) \cap C = A \cap (B \cap C)$.
4. Recall that any rational number can be written as a fraction of integers. That is for any $q \in \mathbf{Q}$, we can write $q = \frac{n}{d}$ where $n, d \in \mathbf{Z}$. Suppose I tried to define a function $f : \mathbf{Q} \to \mathbf{Z}$ by $f\left(\frac{n}{d}\right) = n + d$. Why is such a "function" not well-defined?
5. Recall that if the domain of a function $f : B \to C$ is the same as the codomain of a function $g : A \to B$, we can define the composition of these functions $f \circ g : A \to C$ given by $f \circ g(a) = f(g(a))$.

   (a) Prove that if $f, g : A \to A$ are bijections, then $f \circ g : A \to A$ is a bijection.

   (b) If A is finite with n elements, how many bijections $A \to A$ are there? That is, how many elements are in the set $\mathrm{Bij}(A) := \{f : A \to A \mid f \text{ is bijective}\}$?
6. A relation is a way of comparing two elements in a set. Examples of relations on $\mathbf{Z}$ are $=$, $\leq$, $<$, $\geq$, and $>$. An equivalence relation is a relation $\sim$ that "acts like" equality. Specifically, $\sim$ has the following properties:
   - (Reflexive) $a \sim a$
   - (Symmetric) If $a \sim b$, then $b \sim a$
   - (Transitive) If $a \sim b$ and $b \sim c$, then $a \sim c$.

Prove that each of the following relations are equivalence relations on the given sets:

(a) On the set $\mathbf{Z}$, $n \sim m$ iff $n - m$ is even.

(b) On the set $\mathbf{Z}$, $n \sim m$ iff $n - m$ is divisible by 4.

(c) On the set $\{0, 1, 2, 3, \}$, $n \sim m$ iff $n - m$ is even.

(d) On the set of gumballs in a gumball machine, one gumball is $\sim$ to another iff they are the same color.

7. Given an equivalence relation $\sim$ on a set A and en element $a \in A$, we can define the equivalence class of a by

$$[a] := \{b \in A \,:\, a \sim b\}$$

We can further define the quotient set of A by $\sim$ as the set of equivalence relations. That is

$$A/\sim := \{[a] \,:\, a \in A\}$$

For parts (a), (b), and (c) of exercise 8 above, how many elements are in the respective quotient sets.

8. Given a set A and an equivalence relation $\sim$ on A, prove that the map $p : A \to A/\sim$ given by $p(a) = [a]$ is surjective. This map is called a projection map.