Algebra – Binary Operations & Groups

Advanced Mathematics Program, Summer 2019

For each of the following sets, figure if the proposed operation is a valid (well-defined) binary operation on that set.

1. $(\mathbf{N}, -)$, where $\mathbf{N}$ are the natural numbers, and $-$ is the difference of two numbers.
2. $(\mathbf{Q}, *)$, where $\mathbf{Q}$ is set the rational numbers, and $*$ is the quotient of two numbers. That is $a * b = a/b$.
3. $(X, \star)$, where $X = \{-1, 0, 1, 2, 3\}$ and $a \star b = ab - a - b$.
4. $(W, \times)$, W is the set of all words in the Scrabble dictionary of words, and for any two words $x, y \in W$, we define $x \times y$ to be the word "word".
5. $(X, *)$, where X is a set with three elements, and for $a, b \in X$, $a * b$ will be the element of X that is neither a nor b.

Consider the following binary operation $*$ defined on the set $S = \{a, b, c\}$:

$*$	a	b	c
a	c	a	b
b	a	b	c
c	b	c	a

Prove that this binary operation actually defines a *group* on S. What is the identity element of the group? For each element of S can you identify it's inverse? Finally, can you show that $*$ is associative? (this last part might be tedious)

---

1. For each of the following sets equipped with a binary operation figure why the binary operation *doesn't* give the set a group structure.

   (a) $(\mathbf{N}, \times)$, where $\mathbf{N}$ is the natural numbers, and $\times$ is the usual product of two numbers.

   (b) $(\mathbf{R}, \times)$, where $\mathbf{R}$ is the real numbers, and $\times$ is the usual product of two numbers.

   (c) $(S, *)$, where S be the set of all colors, and $a * b$ is the color that results from mixing the colors a and b in equal parts.

2. Let $\mathbf{Q}^+$ denote the set of positive rational numbers.

(a) Prove that $\mathbf{Q}^+$ under multiplication forms a group.

(b) Define $*$ on $\mathbf{Q}^+$ by $a * b = \frac{1}{2}(a \times b)$. Prove that $(\mathbf{Q}^+, *)$ forms a group.

3. A group $(G, *)$ has to have an identity element e. Prove that this element has to be unique. That is, prove that there cannot be another element $e' \in G$ that satisfied the characterization of an identity element.

4. In a group G, every element has to have an inverse. That is for every group element x there exists a group element x^{-1} such that $xx^{-1} = e$ and $x^{-1}x = e$. Prove that inverse elements are unique. That is, prove that for any x there cannot be more than one element that acts like its inverse.

5. Let $S = \{a, b, c, d\}$. Complete the following table so that $*$ defines a valid *associative* binary operation on S. Is $(S, *)$ a group?

$*$	a	b	c	d
a	a	b	c	d
b	b	a	c	d
c	c	d	c	d
d				

6. Let $S = \{a, b, c, d\}$. After you realize that the S and binary operation in exercise 5 doesn't define a group, can you define an operation on $\star$ on S that *does* define a group? Note that you'll have to define this structure explicitly by writing out the table for the operation $\star$.

7. For a binary operation $*$ on a set S, we'll define an element $x \in S$ to be idempotent if $x * x = x$. Prove that if S is a group, that it has exactly one idempotent element.

8. Consider again the binary operation $\star$ defined above in the first problem on this page, where $\star$ is defined on some set $X \subset \mathbf{Z}$ as $a \star b = ab - a - b$.

   (a) (PLAY) How many *finite* subsets $X \subset \mathbf{Z}$ can you find on which $\star$ will be a valid binary operation on X?

   (b) (MORE PLAY) Prove or disprove: there are infinitely many distinct subsets $X \subset \mathbf{Z}$ on which $\star$ is a valid binary operation.

9. (QUITE TOUGH) Let $*$ be a commutative and associative binary operation on a set S. ($*$ being commutative means that $a * b = b * a$ for every a and b in S.) Assume that for every x and y in S, there exists z in S such that $x * z = y$. (This z may depend on x and y.) Show that if a, b, c are in S and $a * c = b * c$, then $a = b$.