Algebra – Abelian Groups & Subgroups

Advanced Mathematics Program, Summer 2019

Prove that a cyclic group must be abelian.

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1. Let G be a group and H a *nonempty* subset of G. Prove that H is a subgroup of G if and only if for any $a, b \in H$, $ab^{-1} \in H$.

2. Let G be any group and g any element of G. Consider the following set:

$$\langle g \rangle = \{g^k : k \in \mathbf{Z}\}$$

   Prove that $\langle g \rangle$ is a subgroup of G. We call $\langle g \rangle$ the cyclic subgroup of G generated by g.

3. Suppose you have a group G, and two subgroups $H < G$ and $K < G$. Prove that $H \cap K$ will also be a subgroup of G.

4. Prove that any subgroup of an abelian group is abelian.

5. Recall the integers modulo n, $\mathbf{Z}_n$.

   (a) Let $m \in \mathbf{Z}_n$ such that $\gcd(m, n) = 1$. Prove that $\mathbf{Z}_n$ is generated by m.

   (b) Let $m \in \mathbf{Z}_n$ such that $\gcd(m, n) = d$. Prove that the subgroup $\langle m \rangle$ is a group with $\frac{n}{d}$ elements.

6. Prove that any subgroup of a cyclic group must be cyclic.