Algebra – Playing with Groups

Advanced Mathematics Program, Summer 2019

1. Can the empty set be a group?

2. Suppose you have a group with six elements:

$$\{e, a, b, ab, ba, aba\}$$

The element e is the identity element for this group. The element ab, ba, and aba have been given names that indicate how they can be written as a product of the elements a and b. Furthermore these three facts are also true about this group:

$$aa = e \qquad bb = e \qquad aba = bab$$

Can you write out the operation table for this group?

3. Consider the groups whose operation tables are

$+_4$	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

$*$	e	a	b	c
e	e	a	b	c
a	a	b	c	e
b	b	c	e	a
c	c	e	a	b

$*$	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

$\times$	1	-1	i	$-i$
1	1	-1	i	$-i$
-1	-1	1	$-i$	i
i	i	$-i$	-1	1
$-i$	$-i$	i	1	-1

.

You've seen all of these tables this week. Some of these tables actually give the "same" group though.

(a) Which are the same? How many distinct groups are given by those four tables?

(b) (Challenge How many groups are there *total* with four elements? You can figure this out by filling in a general group table an seeing how many choices you had to make, like we did for a group with three elements.

4. Here's a group operation table. Note that in this group, e is not the identity element. Instead a is the identity.

$*$	a	b	c	d	e	f	z	y	x	w	v	u
a	a	b	c	d	e	f	z	y	x	w	v	u
b	b	c	d	e	f	a	y	x	w	v	u	z
c	c	d	e	f	a	b	x	w	v	u	z	y
d	d	e	f	a	b	c	w	v	u	z	y	x
e	e	f	a	b	c	d	v	u	z	y	x	w
f	f	a	b	c	d	e	u	z	y	x	w	v
z	z	u	v	w	x	y	a	f	e	d	c	b
y	y	z	u	v	w	x	b	a	f	e	d	c
x	x	y	z	u	v	w	c	b	a	f	e	d
w	w	x	y	z	u	v	d	c	b	a	f	e
v	v	w	x	y	z	u	e	d	c	b	a	f
u	u	v	w	x	y	z	f	e	d	c	b	a

Isn't that a nice table? Trivia fact: this is the group table for the dihedral group D_6, which one can think of as the group of symmetries of a regular hexagon.

(a) What is the order of the element z? What is the order of the element b?

(b) Can you find a subgroup of this group of order 3?

(c) Is this group commutative?

(d) Identity the subset of elements of this group that commute with *all* the other elements of the group. This subset is called the center of the group.

5. In Exercise 4 we identified the center of a certain group. Let's define the center in general. For a group G, the center of G, often denoted $\mathrm{Z}(G)$, is defined as

$$\mathrm{Z}(G) = \{x \in G : gx = xg \text{ for all } g \in G\}.$$

That is, the center is every element x in G that commutes with *every* other element of G. But the center is not just a subset of the group. Prove that the center is a subgroup of G.

6. Suppose you have a group G, and two subgroups $H < G$ and $K < G$. Prove that $H \cap K$ will also be a subgroup of G.