Algebra – Permutation (Symmetric) Groups

Advanced Mathematics Program, Summer 2019

1. First some healthy computational practice. The group S_3 is the group is the group of all permuations of a set with 3 elements. What is the order of S_3? Can you write down all the elements of S_3?

2. Consider the permutations σ and τ in S_6 given as

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 2 & 5 & 1 & 4 & 6 \end{pmatrix} \qquad \tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 2 & 5 & 1 & 6 & 3 \end{pmatrix}$$

   (a) Can you write down σ^{-1} and τ^{-1}?

   (b) Can you write down $\tau\sigma$ and $\sigma\tau$ and σ^2?

   (c) What is the order of σ?

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3. Using cycle notation, take $\mu = (7\,4\,6) \in S_8$. What is the order of μ? What is the order of the cycle $(1\,5\,2\,4\,8) \in S_8$?

4. Take the same σ and τ above in Exercise 2 and write them in cycle notation. After you've written them in cycle notation, take the products $\sigma\tau$ and $\tau\sigma$ and see that you got the same thing you did before.

5. Say S_n is the group of all permutations of the set $\{1, 2, \ldots, n\}$. Consider the subset of $H_1 \subset S_n$ consisting of all the permutations that leave the number $1 \in \{1, 2, \ldots, n\}$ fixed (all the permutations that send 1 to 1). Prove that H_1 is a *subgroup* of S_n.

6. Prove that the order of S_n is $n!$.

7. Write the following products of cycles as a product of *disjoint* cycles.

$$(9\,8\,7\,6\,5\,4)(1\,5\,9\,4\,3\,8) \qquad (1\,5)(2\,5)(2\,7)(1\,7)(1\,3)(9\,7)(1\,2)(3\,7)$$

8. (Toughie) A transposition is a 2-cycle. So, for example $(2\,3) \in S_3$ is a transposition. Prove that *every* permutation can be we written as a product of transpositions.