Algebra – Quotient Groups and the Isomorphism Theorem

Advanced Mathematics Program, Summer 2019

1. Consider the subgroup $N = \{0, 4, 8\} \leq \mathbf{Z}_{12}$. What does the quotient $\mathbf{Z}_{12}/N$ look like? Based on this example and the example we went over in lecture, what can you conjecture about $\mathbf{Z}_n/N$ where N is some subgroup of $\mathbf{Z}_n$?

2. Let K denote the Klein 4-group and $N = \{e, a\} \trianglelefteq K$. Compute K/N.

3. Recall the group $S_3 = \{(), (12), (23), (13), (123), (132)\}$ and the subgroup $H = \{(), (12)\}$

   (a) Prove that $(123)H \neq H(123)$ and hence H is *not* normal.

   (b) Prove that the binary operation on the cosets of H is *not* well-defined. That is, find $a, b \in S_3$ such that $aH = bH$, but either $axH \neq bxH$ or $xaH \neq xbH$ for some $x \in S_3$.

4. Let Q be the quaternion group and $N = \{1, -1\} \trianglelefteq Q$ a normal subgroup. Compute Q/N.

5. We know that $(\mathbf{R}, +)$ is an abelian groups and so $\mathbf{Z} \subseteq \mathbf{R}$ is a normal subgroup. Prove that every finite cyclic group is isomorphic to a subgroup of $\mathbf{R}/\mathbf{Z}$.

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6. Let $N \trianglelefteq G$ be a normal subgroup and let $\pi\colon G \to G/N$ denote the quotient map. Prove that $\ker(\pi) = N$.

7. Consider the map $f : \mathbf{Z} \to \mathbf{Z}_n$ where $f(m)$ is defined to be the remainder when m is divided by n.

   (a) Prove that this map is a surjective homomorphism.

   (b) Prove that $\ker(f)$ is $n\mathbf{Z}$, the set of integers divisible by n.

   (c) Use the first isomorphism theorem to prove that $\mathbf{Z}_n \cong Z/nZ$.

8. Let $N \trianglelefteq G$ be a normal subgroup and let $\iota\colon N \to G$ denote the inclusion map and $\pi\colon G \to G/N$ denote the quotient map. Prove that

$$\mathrm{Im}(\iota) = \ker(\pi)$$

   where $\mathrm{Im}(\iota)$ denotes the image of the map ι.