AMP - Playing With Groups. Day 4

First, a question:

(1) Can the empty set be a group?

Think about it to yourself to a minute.

... the answer: no it can't be, because a group needs an identity ~~elmt~~ element.

Note quick, another name for "operation tables" is Cayley tables.

(2) Now at the boards in groups, write out the operation table for $G = \{e, a, b, ab, ba, aba\}$ where $aa = e$ $bb = e$ $aba = bab$.

Now you're seen quite a few group operation tables. You may have noticed that in all of them, in any given row or column, there are no duplicate entries. Why is this? How come, if I write down this table for a binary operation $*$ on $\{e, a, b\}$

$*$	e	a	b
e	e	a	b
a	a	~~a~~ e	b
b	b	b	e

you know this won't be a group? Discuss.

ANSWER: Look at those b's I circled. Those tell us that $eb = b$ & $ab = b$. In particular that $eb = ab$, so ~~ab~~ $ebb^{-1} = abb^{-1} \Rightarrow e = a$ ↯.

What if your group operation table looks like this instead?

	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	b
c	c	a	a	e

$$\left.\begin{array}{c} ac = b \\ \text{AND} \\ bc = b \end{array}\right. \Rightarrow ac = bc \Rightarrow acc^{-1} = bcc^{-1}$$

$$\Rightarrow a = b \quad ☹$$

See, the idea is that since every element of a group has an inverse, we have what's called left- and right-cancellation in our group.

$$ax = bx \Rightarrow a = b$$

$$xa = xb \Rightarrow a = b.$$

And as we've seen this implies each row and column of a group operation table cannot have duplicates. This helps us immensely while filling in group operation tables.

Now an idea we've touched on a tiny bit before is that of two different looking groups actually "the same" group. We'll formalize this idea later but the idea is that two groups are "the same" if can just change the names and ~~orders~~ the order the elements are listed ~~in a @ this operation table~~ of one group's operation table to get the other operation table.

Let's play with groups of order 3 as example. Remember the group $\mathbb{Z}_3$?

$+_3$	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

Now I want to fill in a very general group table for a group with three elements $\{e, a, b\}$

$*$	e	a	b
e	e	a	b
a	a	b	e
b	b	e	a

Notice that we didn't make any choices. There was only <u>one</u> way to make that table.

Conclusion: there is really ONLY ONE group with three elements. In particular those ~~two~~ 2 tables give the same group where you just pair up the elements

$$0 \longleftrightarrow e \qquad 1 \longleftrightarrow a \qquad 2 \longleftrightarrow b$$

And, maybe an obvious note, but the order of the rows and columns in the operation table doesn't matter either. This is the same group too: just swap the order of a and e.

	a	e	b
a	b	a	e
e	a	e	b
b	e	b	a

Now that takes care of groups with three elements. What about groups with four elements?

((Give the exercise picking out which groups on four elements are the same.))

Break Time.

More on subgroups, and cyclic subgroups: We talked about this last time but let's rehash. For a group G, <u>any</u> $x \in G$ generates a subgroup of G

$$\langle x \rangle = \{ x^k : k \in \mathbb{Z} \}$$

"Smallest subgroup containing x."

This is the <u>cyclic</u> <u>subgroup</u> of G generated by x.

EXAMPLES:

- $\langle 3 \rangle \overset{\text{Subgroup}}{<} \mathbb{Z}$. $\{\ldots, -9, -6, -3, 0, 3, 6, 9, \ldots\} < \mathbb{Z}$

  more generally $\langle n \rangle < \mathbb{Z}$.

- In $\mathbb{Z}_6 = \{0, 1, 2, 3, 4, 5\}$, you have a subgroup $\{0, 2, 4\} < \mathbb{Z}_6$. This subgroup is both $\langle 2 \rangle$ and $\langle 4 \rangle$. You have another subgroup $\{0, 3\} < \mathbb{Z}_6$ generated by 3.

- On a Rubik's cube, the subgroup of size 4 generated by a single rotating face.

  But not all subgroups are cyclic... : subgroup generated by rotating two faces.

Som terminology that has been overloaded.

The order of a group G, denoted $|G|$, is its size: the number of elements it has (could be ∞).

The order of a subgroup $H < G$ is its size $|H|$.

The order of an element $x < G$ is the size of the cyclic subgroup $\langle x \rangle$ that it generates.

(5) (Then assign the rest of the worksheet. Problem with D_6 doesn't need to be done on the board)

This'll probably take the rest of the time, or the exercise with $Z(G)$, but if not, I could spitball about subgroup lattices.

Offer to give students some nice proof-based exercises over the weekend.