Final Exam

Integral Calculus for Life Sciences

UCR Math-007B-B01, Summer 2019

Name: ____________________________________

Student ID: ____________________________________

Please silence your phone during the exam, and have a photo ID ready for when you submit your exam to me.

You may use the blank side of any page as scratch paper, but note that I will only be looking for you response to a question on the side of the page that contains that question.

Each page of this exam will be weighted equally when being graded.

This exam has many pages. I didn't write it with the expectation that you will finish the whole thing. I doubt anyone will finish it, and that's okay. I want you to pass this course, so I want to provide a wide variety of questions so you can choose which ones you spend time responding to. This way it's okay if there are questions that you don't know how to answer; skip those and move onto questions you *can* respond to.

Remember that the purpose of this exam is to provide me a document to read so that I may assess how much you've learned in the course, and assess how prepared you are for any class for which this one is a prerequisite. Again, I want you to pass this course. I'm not a grading robot, looking only for right answers. I'm just trying to figure out what you know. Give me all the evidence that you can that you've learned the material of this course. Imagine that I'm a lawyer who has to argue to the University for the best grade I can give you with your exam in my hand. Talk to me in your responses on this exam. Even if you are unsure how to perfectly answer a question, tell me what you're thinking about, and tell me what you *do* know that could help answer the question.

1. What is the definition of a *definite integral* of a function on an interval (a, b)? What is the definition of the *indefinite integral* of a function?

2. Write down a calculation to verify that

$$\int \frac{1}{\sin(x)\cos(x)}\,\mathrm{d}x \;=\; \ln(\tan(x)) + C\,.$$

3. Your friend is taking an integral calculus class too. His instructor told the class that for any real numbers a and b,

$$\int_a^b f(x)\,\mathrm{d}x = \int_a^0 f(x)\,\mathrm{d}x + \int_0^b f(x)\,\mathrm{d}x. \qquad (\star)$$

Your friend has good intuition on what a definite integral is, but your friend is confused about why this equation must be true. Explain to your friend why the equation ($\star$) is true. Be sure to consider the case where 0 is not between a and b.

4. Evaluate the following integrals.

(a) $\int x^2 + x + 1 + x^{-1} + x^{-2}\,\mathrm{d}x$

(b) $\int_{-13}^{13} \bigl(2 + f(u)\bigr)\,\mathrm{d}u$, where f is an odd function.

(c) $\int_{0}^{\pi/3} \sec(x)\tan(x)\,\mathrm{d}x$

(d) $\int_{2}^{4} \frac{t - t^2}{t - 1}\,\mathrm{d}t$

(e) $\int_{7}^{7} \mathrm{e}^{\tan(x^2)}\,\mathrm{d}x$

5. Evaluate the following integrals.

(a) $\int \cos(t)e^{2\sin(t)}\,dt$

(b) $\int 2x\sec^2(x)\,dx$

(c) $\int 2\sqrt{1-x^2}\,dx$

6. Consider the function

$$f(x) = \frac{\ln|x^2|}{(x-2)^2}.$$

(a) The integral $\int_0^\infty f(x)\,\mathrm{d}x$ is improper. Write it as a sum of proper integrals with limits.

(b) The integral $\int_0^\infty f(x)\,\mathrm{d}x$ diverges. If I tell you that

$$F(x) = \ln\left|\frac{2-x}{x}\right| - \frac{\ln\left|x^2\right|}{x-2}$$

is an antiderivative of the function f, can you tell me which points in the domain of integration, cause $\int_0^\infty f(x)\,\mathrm{d}x$ to diverge? I.e. around which points in $[0,\infty]$ does $f(x)$ accumulated an infinite area under its graph?

7. What is the definition of a *differential equation*?

8. Verify that $y(x) = e^x \sin(x)$ is a solution to the differential equation $y' - y = \frac{1}{2}y''$.

9. Find *explicit* general solutions to the following differential equations.

(a) $t^2 y'' = 6$

(b) $yy' + y^2 = 3tyy' + 1$

10. Given that $y(3) = 2\mathrm{e}$, find an *explicit* particular solution to the differential equation

$$\frac{y'}{x} = \mathrm{e}^{x-\ln(y^2)}.$$

11. Consider the function $f(t) = t^3 + 2$.

(a) Approximate the value of $\int_0^4 f(t)\,\mathrm{d}t$ by subdividing the domain of integration into four subintervals and using a right-endpoint Riemann sum. Without using the fundamental theorem of calculus, tell me if this approximation is smaller than or larger than the actual value of $\int_0^4 f(t)\,\mathrm{d}t$.

(b) Now *write down* a midpoint Riemann sum using six subintervals that approximates the value of $\int_2^5 f(t)\,\mathrm{d}t$.

12. Let R be the region in the first quadrant bounded by the curves $y = \ln(x)$ and $y = 2$.

(a) Find the volume of the solid that results from rotating the region R about the y-axis.

(b) Write down an integral or sum of integrals that represents the volume of the solid formed by rotating the region R about the x-axis.

(c) Write down an expression that represents the area of the region R.

13. Let $f(x) = 3 + 2x - x^2$, and consider the curve consisting of the points $\big(x, f(x)\big)$.

   (a) Sketch this curve and visually estimate its arclength between $x = 0$ and $x = 3$.

   (b) *Write down* an integral that represents this arclength between $x = 0$ and $x = 3$.

   (c) Approximate the arclength of this curve between $x = 0$ and $x = 3$ using a left-endpoint Riemann sum with three subintervals.

   (d) At what *rate* is the arclength along this curve between $x = 0$ and $x = t$ changing as t varies?

14. The velocity of a car driving around a giant circular track is modelled by the equation $v(t) = t^3 + t$ in miles per hour. Assume the car starts at the starting line at $t = 0$.

  (a) How fast is the car going at $t = 5$ hours.

  (b) How far does the car travel between $t = 0$ and $t = 4$ hours?

  (c) What is the average speed of the car between $t = 0$ and $t = 4$ hours?

15. A population of miniature giraffes in a bio-dome grows at a rate P' modelled by the equation $P'(t) = 500te^{-\frac{1}{5}t^2}$ where $P(t)$ is the population after t years.

(a) If the population right now is 1500, find an equation for $P(t)$ that models what the population will be t years from now.

(b) Is the population of giraffes increasing or decreasing in size? How do you know?

(c) As time goes on indefinitely in the bio-dome, will the size of the giraffe population stabilize? If so, at what size does the population stabilize?

16. Wildlife conservationists want to reintroduce honey badgers to an uninhabited region where they once thrived before being wiped out from inter-badger war. After introducing an initial population of badgers, the conservationists predict that the differential equation

$$\frac{\mathrm{d}P}{\mathrm{d}t} = -\frac{1}{2}\left(P^3 - 4P^2 + 3P\right)$$

will be an accurate model for the population of badgers over time, where $P(t)$ is measured in hundreds of honey badgers after t days of introducing the initial population.

(a) Draw a vector field depicting $\frac{\mathrm{d}P}{\mathrm{d}t}$ for a given value of $P(t)$ and t.

(b) What is the carrying capacity of the population according to this differential equation?

(c) According to the model, how large does the initial population need to be to ensure that the population survives?