Assessment Two

Integral Calculus for Life Sciences

UCR Math-007B-B01, Summer 2019

NAME: Mike Pierce

This assessment is intended not only as a way to determine if you've understood what you were asked to think about for the homework, but also as a means for you to assess your own understanding of that material, and assess if you're meeting your own expectations for yourself. I expect that it'll be challenging for anyone to respond to all these prompts in the allotted time, but that's okay. It wouldn't be a very useful assessment if it wasn't adequately challenging, and furthermore you'd be surprised how much you actually *learn* when you're challenged and under a bit of pressure.

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1. What is a *definite integral*? What is an *indefinite integral*?

A definite integral, denoted $\int_a^b f(x)\,dx$, represents the area under the curve of $f(x)$ as x varies from a to b. You could then think about this "accumulation" of area in terms of what the function $f(x)$ and an antiderivative of $f(x)$ represent.

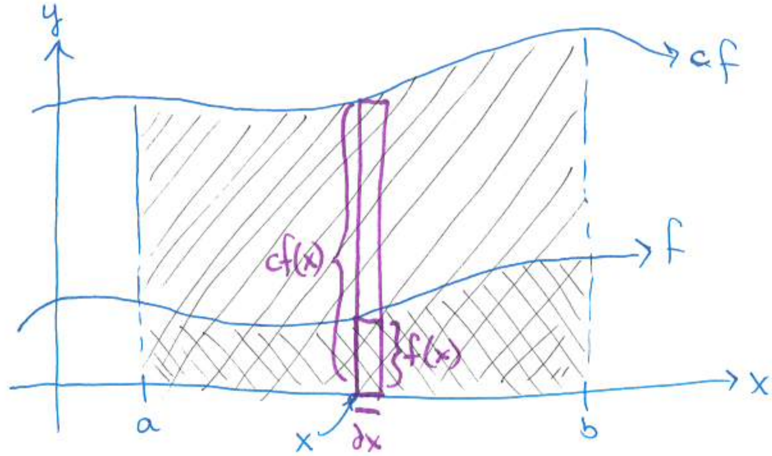
An indefinite integral, denoted $\int f(x)\,dx$ (so without bounds) is the family of antiderivatives of f. So if $\frac{\partial}{\partial x} F(x) = f(x)$, then $\int f(x)\,dx = F(x) + C$.

2. Your friend is taking an integral calculus class too. His instructor told the class that for any real number c,

$$\int_a^b cf(x)\,\mathrm{d}x = c\int_a^b f(x)\,\mathrm{d}x \qquad (\star)$$

Your friend does know how a definite integral can be thought of as an area, but your friend is confused about why this equation must be true. Explain to your friend why the equation $(\star)$ is true.

You've got the two functions cf and f as integrands. Graphing these two function could look something like:



Recall that we find these areas by taking the interval $[a,b]$ dividing it up into subintervals of width dx, and summing the area of a bunch of rectangles. The height of the smaller of these rectangles is $f(x)$ while the height of the taller one is $cf(x)$. Now scaling the height of a rectangle by c ~~is the~~ leads ~~same as~~ to scaling the area of the rectangle by c too.

$$\left((cf(x))dx\right) = c\left(f(x)\,dx\right),$$

So scaling the heights of all the rectangles by c, as in $\int_a^b cf(x)\,dx$, will give us the same result as scaling the area by c, as in $c\int_a^b f(x)\,dx$.

3. Suppose I tell you that

$$\int_0^{\pi/3} \tan(\theta)\,\mathrm{d}\theta = \ln(2) \qquad \text{and} \qquad \int_0^{\pi/3} \sec(\theta)\,\mathrm{d}\theta = \ln\left(2+\sqrt{3}\right).$$

Calculate the values of the following integrals.

(a) $\int_0^{\pi/3} 3\tan(\theta)\,\mathrm{d}\theta$

$= 3(\ln(2)) = \ln(8)$ //

(b) $\int_{\pi/3}^{\pi/3} \tan(\theta)\,\mathrm{d}\theta$ ~~tangent is an odd function, so~~

The bounds are ~~it's zero~~. the same, so there is no area, so it's zero.

0 //

(c) $\int_{\pi/3}^{0} \tan(\theta)\,\mathrm{d}\theta = -\int_0^{\pi/3} \tan(\theta)\,\partial\theta$

$= -\ln(2) = \ln\left(\frac{1}{2}\right)$ //

(d) $\int_{-\pi/3}^{0} \sec(\theta)\,\mathrm{d}\theta = \int_0^{\pi/3} \sec(\theta)\,\partial\theta$

$= \ln(2+\sqrt{3})$ //

since secant is an even function.

(e) $\int_{-\pi/3}^{\pi/3} \tan(\theta)\,\mathrm{d}\theta$ tangent is an odd function, so it's zero.

0 //

(f) $\int_0^{\pi/3} \sec(\theta)+\tan(\theta)\,\mathrm{d}\theta$

$= \int_0^{\pi/3} \sec(\theta)\,\partial\theta + \int_0^{\pi/3} \tan(\theta)\,\partial\theta$

$= \ln(2+\sqrt{3}) + \ln(2)$

$= \ln(4+2\sqrt{3})$ //

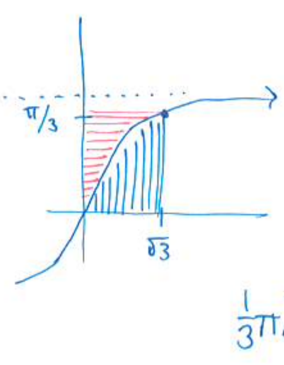
(g) $\int_0^{\pi/3} \sec(\theta)\tan(\theta)\,\mathrm{d}\theta$

$= \sec(\theta)\Big|_0^{\pi/3}$ by FTC

$= \sec\left(\frac{\pi}{3}\right) - \sec(0)$

$= 2 - 1 = 1$ //

(h) $\int_0^{\sqrt{3}} \arctan(x)\,\mathrm{d}x$ arctan(x) is the inverse function of tangent.

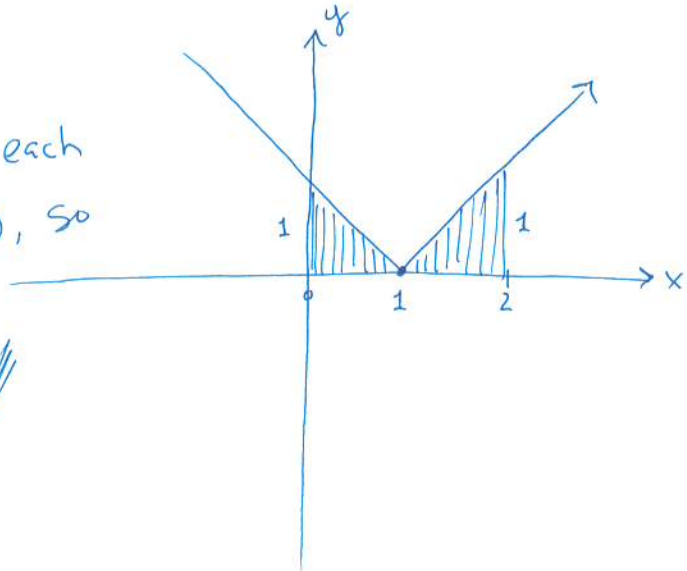


So its equal to our $\frac{\pi}{3} \times \sqrt{3}$ box minus the red area which is equal to $\int_0^{\pi/3} \tan(\theta)\,\partial\theta = \ln(2)$.

$\frac{1}{3}\pi\sqrt{3} - \ln(2)$ //

4. (GEOMETRY) Evaluate the following integrals by looking at the graphs of the integrands and using your knowledge of geometry.

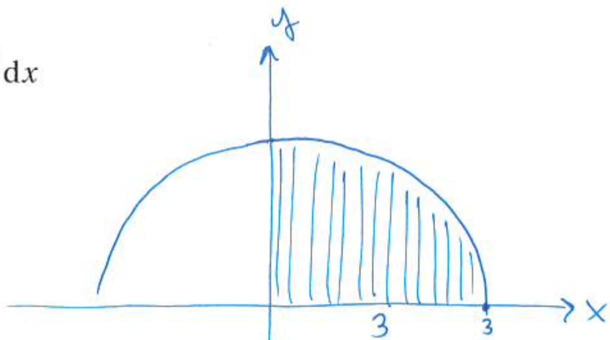
(a) $\displaystyle\int_0^2 |t-1|\,\mathrm{d}t$



It's two triangles, each having area $\frac{1}{2}(1)(1)$, so

$$\int_0^2 |t-1|\,dt = 1$$

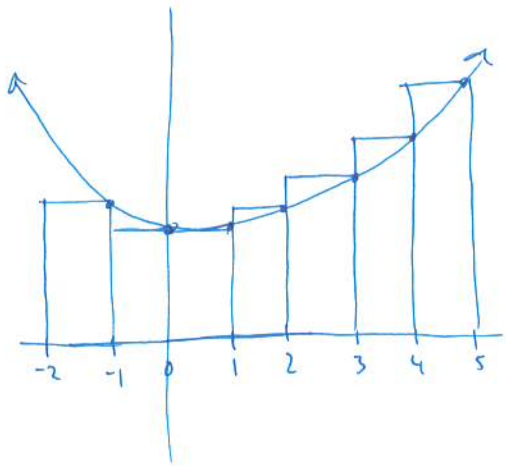
(b) $\displaystyle\int_0^3 \sqrt{9-x^2}\,\mathrm{d}x$



It's a quarter-circle of radius 3, so

$$\int_0^3 \sqrt{9-x^2}\,dx = \frac{1}{4}\left(\pi(3)^2\right) = \frac{9\pi}{4}$$

5. (Riemann Sums) Approximate the value of $\int_{-2}^{5} t^2 + 1\,\mathrm{d}t$ by subdividing the domain of integration into seven subintervals and using a right-endpoint Riemann sum.



The widths are $\frac{5+2}{7} = 1$.

$1(f(-1)) + 1(f(0)) + \cdots + 1(f(5))$

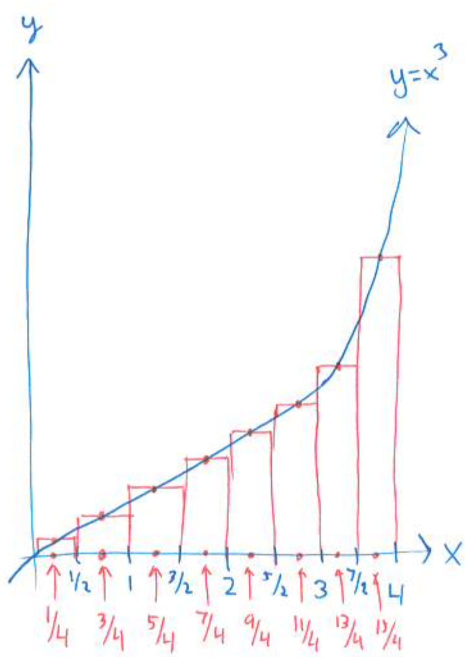
for $f(t) = t^2+1$

$(-1)^2+1 + (0)^2+1 + \cdots + (5)^2+1$

$= (1+0+1+4+9+16+25)+7$

$= (56)+7 = 63$

6. (Riemann Sums) Write down a midpoint Riemann sum that approximates the value of $\int_0^4 \frac{1}{2}x^3\,\mathrm{d}x$ by subdividing the domain of integration into eight subintervals. Notice I'm just asking you to *write down* the sum. It'll probably be gross, so you don't need to *evaluate* the sum.



The width of each rectangle is $\frac{4-0}{8} = \frac{1}{2}$. For $f(x) = \frac{1}{2}x^3$,

$\frac{1}{2}f(1/4) + 1/2\,f(3/4) + \cdots + 1/2\,f(15/4)$

$= \frac{1}{2}\left(\frac{1}{2}(1/4)^2 + 1/2(3/4)^2 + \cdots + 1/2(15/4)^2\right)$

$= \frac{1}{64}\left(1+3+25+49+81+121+169+225\right)$

7. (CALCULATIONS) Evaluate the following integrals.

(a) $\displaystyle\int_{-1}^{3} x^2 \,\mathrm{d}x$ $= \frac{1}{3}x^3 \Big|_{-1}^{3} = \frac{1}{3}\left(3^3 - (-1)^3\right)$

$= \frac{1}{3}\left(27 + 1\right) = \frac{28}{3}$

(b) $\displaystyle\int_{\mathrm{e}}^{5} \frac{1}{t} - \frac{1}{5} \,\mathrm{d}t$ $= \left(\ln(t) - \frac{1}{5}t\right)\Big|_{e}^{5}$

$= \left(\ln(5) - \frac{1}{5}(5)\right) - \left(\ln(e) - \frac{1}{5}(e)\right)$

$= \ln(5) - 1 - 1 + \cancel{1}\,\frac{e}{5} = \ln(5) + 2 + \frac{e}{5}$

(c) $\displaystyle\int 3\cos(t) + \frac{1}{2}\csc^2(t) \,\mathrm{d}t$

$= 3\sin(t) \cancel{+} - \frac{1}{2}\cot(t) + C$

(d) $\displaystyle\int \frac{2x^2 - x}{\sqrt{x}} \,\mathrm{d}x$ $= \int 2x^{3/2} - x^{1/2} \,\partial x = 2\int x^{3/2} \partial x - \int x^{1/2} \partial x$

$= 2\left(\frac{2}{5}\right)x^{5/2} - \frac{2}{3}x^{3/2} + C$

$= \frac{4}{5}x^{5/2} - \frac{2}{3}x^{3/2} + C$

8. Using the first part of the fundamental theorem of calculus, take the following derivative. HINT: Remember the chain rule.

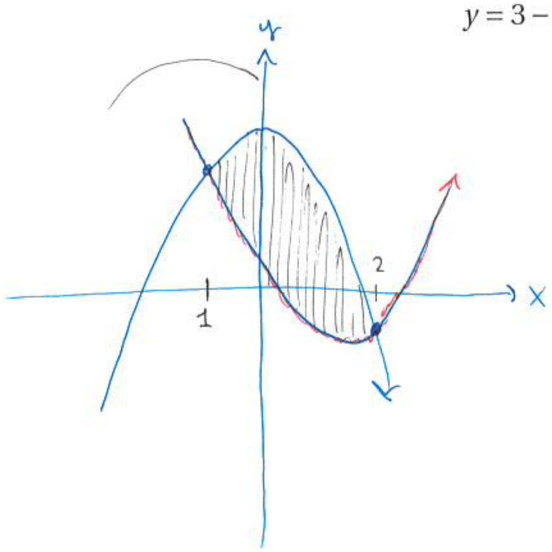
$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{t^3}^{0} \cot(x)\cos(x)\,\mathrm{d}x$$

$$= -\frac{\partial}{\partial t}\int_{0}^{t^3} \cot(x)\cos(x)\,\partial x$$

$$= -\cot\left(t^3\right)\cos\left(t^3\right)\left(3t^2\right)$$

9. Find the area of the region in the (x, y)-plane bounded by the curves

$$y = 3 - x^2 \qquad \text{and} \qquad y = x^2 - 6x + 7.$$



it's an upward facing parabola. Who know's how to graph it, but if there is a region at all, it'll intersect $y = 3 - x^2$ in two spots.

$$3 - x^2 = x^2 - 6x + 7$$

$$\Longrightarrow \quad 0 = 2x^2 - 6x + 4$$

$$\Longrightarrow \quad 0 = 2(x-2)(x-1)$$

So the drawing isn't perfect, but we know that the graphs intersect where $x = 1$ and $x = 2$, so the area is

$$\int_1^2 (3 - x^2) - (x^2 - 6x + 7)\,dx = \int_1^2 -2x^2 + 6x - 4\,dx$$

$$= -2\left(\frac{1}{3}x^3 - \frac{3}{2}x^2 + 4x\right)\Big|_1^2 = -2\left(\frac{8}{3} - 6 + 8 - \frac{1}{3} + \frac{3}{2} - 4\right)$$

$= \dots$ gross :)

10. (RECREATION) You're the director at an airport on the equator, and each plane at your airport can hold exactly enough fuel to fly half way around the world. The planes can also meet up in midair to share fuel. Your task is to use as few planes as possible to successfully fly one plane all the way around the world, but all planes that you use must safely land back at your airport. How many planes do you need?

Talk to Mike