Assessment Four

Integral Calculus for Life Sciences

UCR Math-007B-B01, Summer 2019

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This assessment is intended not only as a way to determine if you've understood what you were asked to think about for the homework, but also as a means for you to assess your own understanding of that material, and assess if you're meeting your own expectations for yourself. I expect that it'll be challenging for anyone to respond to all these prompts in the allotted time, but that's okay. It wouldn't be a very useful assessment if it wasn't adequately challenging, and furthermore you'd be surprised how much you actually *learn* when you're challenged and under a bit of pressure.

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1. Your friend is taking an integral calculus class too. His instructor told the class that when you evaluate an indefinite integral, you have to put a $+C$ at the end of your antiderivative. So if f is the derivative of F,

$$\int f(x)\,\mathrm{d}x = F(x) + C$$

Your friend doesn't know why we do this, and is also a bit uncertain what an *indefinite integral* even is. Explain to your friend what an indefinite integral is, and why you have to add a $+C$.

An indefinite integral is a family of antiderivatives. For any antiderivative F of f, $F(x)+C$ will also be an antiderivative since $\frac{d}{dx}\left(F(x)+C\right) = \frac{d}{dx}F(x) + 0 = f(x)$. The $+C$ represents the arbitrary constant summand on an antiderivative of f. Or, if you'd like, define $\int f(x)\,dx$ specifically to be the set of all functions $\int_a^x f(t)\,dt$ for some constant $\underline{a}$. Then $F(a) = C$.

2. Evaluate each of the following integrals.

(a) $\int_{-999}^{999} 999\mu^{999}\, \mathrm{d}\mu$ $= 0$, since μ^{999} is odd. //

(b) $\int 3t^2 + 2t + 1 + \frac{1}{t} - \frac{1}{t^2}\, \mathrm{d}t$ $= t^3 + t^2 + t + \ln(t) + \frac{1}{t} + C$ //

(c) $\int_1^2 \sqrt{\frac{(x^2-1)(x-1)}{x+1}}\, \mathrm{d}x$ $= \int_1^2 \sqrt{\frac{\cancel{(x+1)}(x-1)(x-1)}{\cancel{x+1}}}\, dx$

$\dots = \int_1^2 \sqrt{(x-1)^2}\, dx = \int_1^2 x - 1\, dx = \frac{1}{2}x^2 - x \Big|_1^2 = \frac{1}{2}$ //

(d) $\int_0^{\pi/3} \sec^2(\theta) e^{\tan(\theta)}\, \mathrm{d}\theta$ Let $u = \tan(\theta)$, so $du = \sec^2(\theta)\, d\theta$.

$= \int_{\tan(0)}^{\tan(\pi/3)} e^u\, du = \int_0^{\sqrt{3}} e^u\, du = e^u \Big|_0^{\sqrt{3}} = e^{\sqrt{3}} - 1$ //

3. Write down two distinct antiderivatives of the function $f(x) = 3x^2 - 1$.

$x^3 - x + C$ for any number C will do. For example:

$x^3 - x + \pi$

$x^3 - x - 17$

$x^3 - x + 37/e \ldots$

4. Which one of the following functions is an antiderivative of $f(x) = \tan^3\theta$?
HINT: Remember that $\tan^2\theta = \sec^2\theta - 1$.

$F(\theta) = \frac{1}{2}\sec^2\theta + \ln\left(\frac{1}{2}\cos\theta\right)$ $\qquad$ $F(\theta) = \frac{1}{2}\tan^4\theta - \sec\theta + 1$

Because the derivative of this one is $F'(\theta) = \tan^3(\theta)$.

$$F'(\theta) = \frac{1}{2}(2\sec\theta)(\sec\theta\tan\theta) + \frac{1}{\frac{1}{2}\cos\theta}\left(\frac{1}{2}(-\sin\theta)\right)$$

$$= \sec^2\theta\tan\theta + \left(-\frac{\sin\theta}{\cos\theta}\right)$$

$$= \tan\theta(\sec^2\theta - 1)$$

$$= \tan^3\theta$$

Whereas if you look at the derivative of this one, you get

$$2\tan^3\theta\sec^2\theta - \sec\theta\tan\theta$$

$$= \sec\theta\tan\theta(2\sec\theta\tan^2\theta - 1)$$

$$= \ldots$$

Because of that $\sec^1\theta$, you can never get just $\tan^3\theta$.

5. Here are two integrals. Choose *one* of them and evaluate it by whatever method you find that works. Please circle the one you choose.

$$\int t^2 e^{-t}\,dt \qquad\qquad \int x^2 \sin(x)\,dx$$

Both of these integrals suggest integration by parts.

$u = t^2 \quad dv = e^{-t}\,dt$

$du = 2t\,dt \quad v = -e^{-t}$

$= -t^2e^{-t} - \int 2t(-e^{-t})\,dt$

$= -t^2e^{-t} + 2\int te^{-t}\,dt$

$u = t \quad dv = e^{-t}\,dt$

$du = dt \quad v = -e^{-t}$

$= -t^2e^{-t} + 2\left(-e^{-t}t - \int -e^{-t}\,dt\right)$

$= -t^2e^{-t} - 2te^{-t} + 2\int e^{-t}\,dt$

$= -t^2e^{-t} - 2te^{-t} - 2e^{-t} + C$

$u = x^2 \quad dv = \sin(x)\,dx$

$du = 2x\,dx \quad v = -\cos(x)$

$= -x^2\cos(x) - \int 2x(-\cos(x))\,dx$

$= -x^2\cos(x) + 2\int x\cos(x)\,dx$

$u = x \quad dv = \cos(x)\,dx$

$du = dx \quad v = \sin(x)$

$= -x^2\cos(x) + 2\left(x\sin(x) - \int \sin(x)\,dx\right)$

$= -x^2\cos(x) + 2x\sin(x) + 2\cos(x) + C$

6. Here are three integrals. Choose *one* of them and evaluate it by whatever method you find that works. Please circle the one you choose.

(A) $\displaystyle\int \frac{x}{x^2+1}\,dx$ (B) $\displaystyle\int \frac{x^2}{x^2+1}\,dx$ (C) $\displaystyle\int \frac{x^3}{x^2+1}\,dx$

(A) Let $u = x^2+1$, so $du = 2x\,dx$ (so $\frac{1}{2}du = x\,dx$)

$$\int \frac{x}{x^2+1}\,dx = \frac{1}{2}\int \frac{1}{u}\,du = \frac{1}{2}\ln(u) + C = \ln\left(\sqrt{x^2+1}\right) + C$$

(B) Remember that trick

$$\int \frac{x^2}{x^2+1}\,dx = \int \frac{x^2+1-1}{x^2+1}\,dx = \int 1 - \frac{1}{x^2+1}\,dx$$

$$= x - \arctan(x) + C$$

(C)

$$\begin{array}{r} x \\ x^2+1 \,\big)\, \overline{x^3 + 0x^2 + 0x + 0} \\ \underline{-x^3 - x} \\ -x + 0 \end{array}$$

$$\int \frac{x^3}{x^2+1}\,dx = \int x - \frac{x}{x^2+1}\,dx$$

$$= \frac{1}{2}x^2 - \int \frac{x}{x^2+1}\,dx$$

Which is just part (A) above.

$$= \frac{1}{2}x^2 - \ln\left(\sqrt{x^2+1}\right) + C$$

7. Here are two integrals. Choose *one* of them and evaluate it by whatever method you find that works. Please circle the one you choose.

$$\int \frac{x^3+x^2-1}{x^2-x}\,dx \qquad\qquad \int \frac{3x^2+14x+9}{x(x+3)^2}\,dx$$

The degree in the numerator is larger than the degree in the denominator.

$$\begin{array}{r} x+2 \\ x^2-x \,\big)\, \overline{x^3+x^2+0x-1} \\ \underline{-x^3+x^2 } \\ 2x^2 \\ \underline{-2x^2+2x } \\ 2x-1 \end{array}$$

$$\int \frac{x^3+x^2-1}{x^2-x}dx = \int x+2+\frac{2x-1}{x^2-x}\,dx$$

$$= \frac{1}{2}x^2+2x+\int\frac{A}{x}dx+\int\frac{B}{x-1}dx$$

$$2x-1 = A(x-1)+Bx$$
$$= (A+B)x+(-A)$$

So $A=1$ and $B=1$

$$\ldots = \frac{1}{2}x^2+2x+\int\frac{1}{x}dx+\int\frac{1}{x-1}dx$$

$$= \frac{1}{2}x^2+2x+\ln(x)+\ln(x-1)+C$$

//

$$= \int \frac{A}{x}+\frac{B}{x+3}+\frac{C}{(x+3)^2}\,dx$$

where we dream that

$$3x^2+14x+9 = A(x+3)^2+B(x)(x+3)+Cx$$
$$= Ax^2+6Ax+9A+Bx^2+3Bx+Cx$$
$$= (A+B)x^2+(6A+3B+C)x+9A$$

$$\begin{cases} 3 = A+B \\ 14 = 6A+3B+C \\ 9 = 9A \end{cases} \Rightarrow \begin{array}{l} A=1 \\ B=2 \\ C=2 \end{array}$$

$$= \int\frac{1}{x}dx+\int\frac{2}{x+3}dx+\int\frac{2}{(x+3)^2}dx$$

$$= \ln(x)+2\ln(x+3)-\frac{2}{x+3}+C$$

//

8. Here are three integrals. Choose *one* of them and evaluate it by whatever method you find that works. Please circle the one you choose.

(A) $\displaystyle\int \frac{\ln(\ln(x))}{x\ln(x)}\,dx$ (B) $\displaystyle\int \frac{dx}{1+e^x}$ (C) $\displaystyle\int \cancel{x(\ln(x))^2\,dx}$

(A) Let $u=\ln(x)$, so $du=\frac{1}{x}\,dx$. $\displaystyle\int \frac{\ln(\ln(x))}{x\ln(x)}\,dx=\int \frac{\ln(u)}{u}\,du$.

Let $w=\ln(u)$, so $dw=\frac{1}{u}\,du$. $\displaystyle\int \frac{\ln(u)}{u}\,du=\int w\,dw=\ldots$

$\ldots=\frac{1}{2}w^2+C=\frac{1}{2}\ln(u)^2+C=\ln\left(\sqrt{\ln(x)}\right)^2+C$ //

(B) Let $u=1+e^x$, so $du=e^x\,dx=(u-1)\,dx$ (so $\frac{du}{u-1}=dx$)

$$\int \frac{dx}{1+e^x}=\int \frac{du}{u(u-1)}=\int \frac{A}{u}\,du+\int \frac{B}{u-1}\,du$$

We need

$0u+1=A(u-1)+B(u)$

$\Rightarrow \begin{cases} A+B=0 \\ -A=1 \end{cases}$

So $A=-1$ and $B=1$

$$\ldots=-\int \frac{1}{u}\,du+\int \frac{1}{u-1}\,du=-\ln(u)+\ln(u-1)+\cancel{C}^{\ln(c)}$$

$$=\ln\left(c\,\frac{u-1}{u}\right)$$ // if you prefer

(C) This one will yield to a substitution $u=\ln(x)$, then integration by parts twice.

OR you could use "by parts" first for $u=x\ln(x)^2$, $dv=dx$, and get it eventually that way. Still, too tough.

9. An oil tanker hits a reef hidden beneath the ocean's surface and begins leaking oil at a rate (in barrels per hour) modeled by the equation

$$L'(t) = \frac{80\ln(t+1)}{t+1}$$

where t is measured in hours since the tanker hit the reef. How many barrels of oil will the tanker lose on the first day? How many barrels of oil will the tanker lose on the second day?

First day means the first 24 hrs.

$$\int_0^{24} \frac{80\ln(t+1)}{t+1}\,dt \qquad \text{Let } u = t+1,\ du = dt.$$

$$= 80\int_1^{25} \frac{\ln(u)}{u}\,dt \qquad \text{Let } w = \ln(u),\ dw = 1/u\,du$$

$$= 80\int_0^{\ln(25)} w\,dw = 80\left(\frac{1}{2}w^2\right)\Big|_0^{\ln(25)} = 40\ln(25)^2$$

$$= 160\ln(5)^2 \text{ if you prefer.}$$

Second day means hours 24 to 48

$$\int_{24}^{48} \frac{80\ln(t+1)}{t+1}\,dt = 80\int_{\ln(25)}^{\ln(49)} w\,dw$$

$$= 80\left(\frac{1}{2}w^2\right)\Big|_{\ln(25)}^{\ln(49)} = 40\left(\ln(49)^2 - \ln(25)^2\right)$$

$$= 160\ln\left(\frac{7}{5}\right)\ln(35) \text{ if you prefer.}$$

10. (RECREATIONAL) Five women and a monkey were shipwrecked on a desert island. They spent the first day gathering coconuts for food, piling up all the coconuts together before they went to sleep for the night. But while they were all asleep, one woman woke up and thought there might be a row about dividing the coconuts in the morning, so she decided to get up and take her share now. She divided the coconuts into five equal piles except for one coconut left over which she gave to the monkey, and she hid her pile and put the rest back together. By and by, another woman woke up and did the same thing, and similarly she had one coconut left over which she gave to the monkey. And each of the five women did the same thing, one after the other; each one taking a fifth of the coconuts in the pile when she woke up, and each one having one left over for the monkey. In the morning they divided what coconuts were left, and they came out in five equal shares. Of course each one must have known that there were coconuts missing, but each one was as guilty as the others, so they didn't say anything. How many coconuts were there in the beginning?

See Mike :)