Homework Four

Integral Calculus for Life Sciences
UCR Math-007B-B01, Summer 2019

1. (Trigonometric Integrals) Evaluate each of the following integrals. Solutions to a couple of them can be found in Paul's Online Notes.

$$\int \sin^2(x)\cos^7(x)\,\mathrm{d}x \qquad \int \sec^3(x)\tan^4(x)\,\mathrm{d}x \qquad \int \cos^4(\theta)\,\mathrm{d}\theta$$

$$\int \tan^3(\mu)\,\mathrm{d}\mu \qquad \int \cot(10z)\csc^4(10z) \qquad \int \frac{2+7\sin^3(x)}{\cos^2(x)}\,\mathrm{d}x$$

2. (Trigonometric Integrals) For each of the following forms of integral, explain how you can evaluate it for the different cases of values of m and n. For example, "If both n and m are odd, you do this thing. But if n is even and m is odd, you do this thing instead. But if ..."

$$\int \sin^n(x)\cos^m(x)\,\mathrm{d}x \qquad \int \sec^n(x)\tan^m(x)\,\mathrm{d}x$$

3. (Trigonometric Substitution) Evaluate each of the following integrals.

$$\int \frac{\mathrm{d}x}{1+x^2} \qquad \int \frac{\mathrm{d}x}{1-x^2} \qquad \int \frac{\mathrm{d}x}{\sqrt{1-x^2}}$$

$$\int \frac{\mathrm{d}x}{\sqrt{1+x^2}} \qquad \int \frac{\mathrm{d}x}{\sqrt{x^2-1}} \qquad \int \frac{\mathrm{d}x}{x\sqrt{x^2-1}}$$

$$\int \frac{\mathrm{d}x}{x\sqrt{1-x^2}} \qquad \int \frac{\mathrm{d}x}{x\sqrt{1+x^2}} \qquad \int x^3\sqrt{1-x^2}\,\mathrm{d}x$$

$$\int \sqrt{1-x^2}\,\mathrm{d}x \qquad \int \sqrt{1+x^2}\,\mathrm{d}x \qquad \int \sqrt{x^2-1}\,\mathrm{d}x$$

4. (Trigonometric Substitution) Evaluate each of the following integrals. Solutions to these are written up in Paul's Online Math Notes.

$$\int \sqrt{1-7\omega^2}\,\mathrm{d}\omega \qquad \int_{-7}^{-5} \frac{2\,\mathrm{d}y}{y^4\sqrt{y^2-25}} \qquad \int \frac{\mathrm{d}x}{\sqrt{9x^2-36x+37}}$$

5. Evaluate each of the following integrals. Completing the square in the denominator first might be a good idea.

$$\int \frac{1}{x^2-2x+1}\,\mathrm{d}x \qquad \int \frac{1}{x^2+2x+5}\,\mathrm{d}x \qquad \int \frac{1}{x^2-16x+64}\,\mathrm{d}x$$

6. (IMPROPER INTEGRALS) Evaluate each of the following improper integrals. In this case, "evaluate" means to determine if the integral has a value (if the implied limits exist), and if it has a value, what is that value. Some of these I borrowed from Paul's Online Math Notes, so you can find some solutions there. More exercises like this can be found on the internet or in Neuhauser Chapter 7.4.

$$\int_0^\infty xe^{-x}\,\mathrm{d}x \qquad \int_e^\infty \frac{\mathrm{d}x}{x\ln(x)^2} \qquad \int_0^{\pi/2} \frac{\cos(x)}{\sqrt{\sin(x)}}\,\mathrm{d}x$$

$$\int_{-5}^1 \frac{1}{10+2z}\,\mathrm{d}z \qquad \int_0^4 \frac{x}{x^2-9}\,\mathrm{d}x \qquad \int_{-\infty}^\infty \frac{6w^3}{(w^4+1)^2}\,\mathrm{d}w$$

$$\int_{-1}^1 \ln|x|\,\mathrm{d}x \qquad \int_1^\infty \frac{1}{x^3}\,\mathrm{d}x \qquad \int_1^\infty \frac{1}{x^{1/3}}\,\mathrm{d}x$$

$$\int_{-\infty}^1 \frac{3}{1+x^2}\,\mathrm{d}x \qquad \int_{-\infty}^0 \frac{e^{\frac{1}{x}}}{x^2}\,\mathrm{d}x \qquad \int_0^2 \frac{1}{(x-1)^4}\,\mathrm{d}x$$

7. (IMPROPER INTEGRALS) Let's define the function $A(z) = \int_1^z x^{-p}\,\mathrm{d}x$ for $z > 1$.

   (a) Show that if $p = 1$, then $A(z) = \ln|z|$, but that otherwise, for $p \neq 1$, we have

   $$A(z) = \frac{1}{1-p}\left(z^{1-p}-1\right).$$

   (b) Show that if $p \in (0,1]$, then $\lim_{z\to\infty} A(z) = \infty$.

   (c) Now show that if $p > 1$, then $\lim_{z\to\infty} A(z) = \frac{1}{p-1}$.

8. (MICROTUTORIAL) The rate r at which people get sick during a flu epidemic can be modelled by the function

$$r(t) = 1600te^{-\frac{1}{5}t}$$

where r is measured in people per day and t is measured in days since the start of the epidemic. Sketch a graph of r as a function of t. At what day since the start of the epidemic are people getting sick the fastest? How many people get sick over the course of the entire epidemic? To see a similar example, watch blendedmathematics.ucr.edu/spreading_the_flu_2.html

9. (MICROTUTORIAL) An oil tanker hits a reef hidden beneath the ocean's surface. The oil storage compartments have been breached, and considering the pressure in the tanks, the tanker begins leaking oil at a rate (in barrels per hour) modelled by the equation

$$L'(t) = \frac{80\ln(t+1)}{t+1}$$

where t is measured in hours since the tanker hit the reef. If the tanker is bound to leak *all* of its oil if the leak continues indefinitely how much oil was the tanker carrying in the first place? To see some of this example worked through, watch

blendedmathematics.ucr.edu/oil_leak.html

10. (MICROTUTORIAL) A charged wire creates an electric field at a point P located at a distance D from the wire. The component $E_\perp$ of the field perpendicular to the wire (in newtons per coulomb) is

$$E_\perp = \int_{x_1}^{x_2} \frac{\mathrm{k}\lambda D}{(x^2+D^2)^{3/2}}\,\mathrm{d}x$$

where λ is the charge density (coulombs per meter), k is the Coulomb constant, and x_1 andx_2 are two points on the wire to the left and right of D respectively. First write $E_\perp$ simply as a function of x. What happens as the distance D increases to infinity? What happens if you consider P located centrally between x_1 and x_2 moving infinitely apart? In other words, calculate $E_\perp$ for an infinitely long wire where P is a distance D above the center of the wire. To see this exercise worked through, watch

blendedmathematics.ucr.edu/electric_field.html

11. (TOUGH) Evaluate the following integrals.

$$\int \frac{\mathrm{d}x}{\sqrt{1+\mathrm{e}^{2x}}} \qquad \int \ln\left(\mu+\sqrt{1-\mu}\right)\,\mathrm{d}\mu \qquad \int \frac{\mathrm{d}\theta}{1-\sin^4(\theta)}$$

12. (CHALLENGE) Evaluate the following integrals.

$$\int \frac{\mathrm{d}\omega}{\left(\omega^2+1\right)\sqrt{\omega^2-1}} \qquad \int \sqrt{\tan(\theta)}\,\mathrm{d}\theta \qquad \int_0^{\pi/2} \ln(\cos(t))\,\mathrm{d}t$$

13. (CHALLENGE: THE $x \leftrightarrow \frac{1}{x}$ TRICK) Show that the integral

$$\int_0^\infty \frac{\ln(x)}{1+x^2}\,\mathrm{d}x$$

evaluates to zero by breaking up its domain of integration into two parts based on where the integrand is negative or positive, and using the substitution $x \leftrightarrow \frac{1}{x}$ on one of those parts. Then use this same trick to show that the value of the following integral doesn't depend at all on the value of the real number a.

$$\int_0^{\pi/2} \frac{\mathrm{d}\theta}{1+\big(\tan(\theta)\big)^a}$$

For some helpful reading related to this trick, see

math.stackexchange.com/q/2060187.

14. (CHALLENGE, SPIVAK) Consider the curve given by $y = \frac{1}{x}$ for $x \geq 1$. This shape is sometimes called Gabriel's horn.

    (a) What is the volume of the inside of the horn?

    (b) You can find the surface area of the horn by taking the curve $y = \frac{1}{x}$, writing down a formula for it's arclength, and rotating that arclength about the x-axis over a bunch of small subintervals, and taking an integral, thereby summing up a lot of smaller surface areas. Do this, and show that the horn has infinite surface area.

    (c) Suppose you take a volume of paint that is equal to the volume of the inside of the horn and pour it into the horn. This would seem to paint the entire inside of the horn, but we will have painted the infinite surface of the horn with finitely much paint. How can this be?

15. (CHALLENGE: 2005 PUTNAM EXAM) Evaluate

$$\int_0^1 \frac{\ln(x+1)}{1+x^2}\,\mathrm{d}x.$$

16. (CHALLENGE: 1987 PUTNAM EXAM) Evaluate

$$\int_2^4 \frac{\sqrt{\ln(9-x)}}{\sqrt{\ln(9-x)}+\sqrt{\ln(x+3)}}\,\mathrm{d}x.$$