Homework Five

Integral Calculus for Life Sciences

UCR Math-007B-B01, Summer 2019

1. What is the definition of a differential equation? Be sure to know a definition that you would be proud to write down if you were asked this question on an exam.

2. Verify that each of the following is a solution to the given differential equation.

   (a) Verify that $p(t) = 3e^{kt}$ is a solution to $p' = kp$.

   (b) Verify that $y = 2t^3$ is a solution to $3yy''' = y'y''$.

   (c) Verify that $y(t) = \cos(2t)$ is a solution to $yy' + \sin(4t) = 0$.

   (d) Verify that $y(t) = \cos(2t)$ is also a solution to $yy'' + 4 = (y')^2$.

3. Suppose that
$$y' = 2t^3 + \frac{1}{7t} - \sin(2t)\,.$$
Write down the *general solution* for y as a function of t. (HINT: remember that this is just another way of phrasing a question you've been asked many times in this class already.) What must y be *in particular* if we know that $y = \frac{1}{2}\cos(2)$ when $t = 1$?

4. Suppose that
$$y'' = x^2 + 3\,.$$
but we know that $y = -4$ and $y' = 2$ when $x = 0$. What must y be?

5. Solve these differential equations. (HINT: Notice that I'm not providing any initial conditions, so I'm asking for the *general* solution to each of these.)

   $2\cos(t) = 3t^2 - y'$ $\qquad$ $y' - 6y = 4$ $\qquad$ $xy' - 2y' = 2(y-4)$

6. Solve these differential equations. (HINT: Notice that I'm providing you with initial conditions for these differential equations, so these are examples of *initial value problems*, and I'm asking for a *particular* solution to each of these.)

   $\frac{1}{t}\sec^2(y)\frac{dy}{dt} = 1$ where $y(0) = 1$. $\qquad$ $e^t - yy' = 0$ where $y(0) = 3$.

7. (MICROTUTORIAL) Suppose that a fish population in a lake can be accurately modelled using the logistic equation, and that the fish are harvested at a rate $h > 0$ (per day) proportional to the population size. If $N(t)$ denotes the population size at time t measured in days, then this equation is

$$\frac{\mathrm{d}N}{\mathrm{d}t} = rN\left(1 - \frac{N}{E}\right) - hN$$

where r is the naïve rate (per day) at which the population increases proportionate to its size, and E is the carrying capacity of the population (without human interaction). Through previous studies of this species of fish, we know that $r \approx 5$, and we know that the carrying capacity of the lake is about 1000 fish, based on the resources available to the fish in the lake.

(a) Graph the vector field of this differential equation for a harvest rate $h = 1$. For this value of h, what are the equilibrium solutions? Interpreting this result, if we remove one fish from the lake per day, what is the long-term expected size of the fish population?

(b) Algebraically determine all the equilibrium solutions to this differential equation as a function of h. For which values of h do we have a nontrivial equilibrium solution? Interpreting this result, what is the maximum number of fish that we may remove from the lake per day to guarantee that the population doesn't die out?

For a walk-through of a question similar to this one, see:

blendedmathematics.ucr.edu/fish_population.html

8. Some wildlife conservationists want to reintroduce flamingos to an uninhabited region where flamingos once thrived before being wiped out from excessive hunting. After introducing an initial population of flamingos to the region, the conservationists suspect that the differential equation

$$\frac{\mathrm{d}P}{\mathrm{d}t} = -\frac{1}{2}\left(P^3 - 4P^2 + 3P\right)$$

will be an accurate model for the population of flamingos over time, where $P(t)$ is measured in thousands of flamingos after t years of introducing the flamingos. What is the *carrying capacity* of the population according to this differential equation? According to the model, how large does the initial population need to be to ensure that the population survives? (HINT: consider the vector field.)

9. (MICROTUTORIAL) A drug has first-order elimination kinetics, meaning that a fixed fraction of drug is eliminated from the body per hour. So if no further drug is absorbed into the patient's blood after time $t = 0$, then the amount of drug M in their blood, measured in milligrams, will decay with time according to the differential equation

$$\frac{\mathrm{d}M}{\mathrm{d}t} = -kM$$

where $k > 0$ is the fraction of drug eliminated in one hour, and time is measured in hours.

(a) Assuming a certain patient starts with an amount of drug in their body equal to M_0 milligrams, write down an equation $M(t)$ that models the amount of drug remaining in their bloodstream after t hours.

(b) According to this model, does the drug ever vanish completely from the patient's bloodstream?

(c) If the patient started with 10 milligrams of the drug in their bloodstream, and studies have shown that $k \approx 2$ for this specific drug, in how many hours will the amount of drug remaining in their bloodstream be about 1 milligram?

For a walk-through of a question similar to this one, see:

blendedmathematics.ucr.edu/insulin_pump.html