IntBioCalc Week 1 Monday

Let's start this class with a motivating example.

QUESTION — A car begins moving at $t=0$ with a constant velocity ~~given by the~~ of 30 MPH. How far has the car moved in 5 hours?

Remember that (rate) x (time) = distance. If you don't remember this, just know that you need an equation with these three quantities such that the units ~~xxxx~~ work out:

$$\left(\frac{\text{miles}}{\text{hour}}\right) \times (\text{hours}) = \text{miles}$$

$$\text{rate} \times \text{time} = \text{distance}$$

So the car travelled a total of (30 MPH)(5 hours) = ...

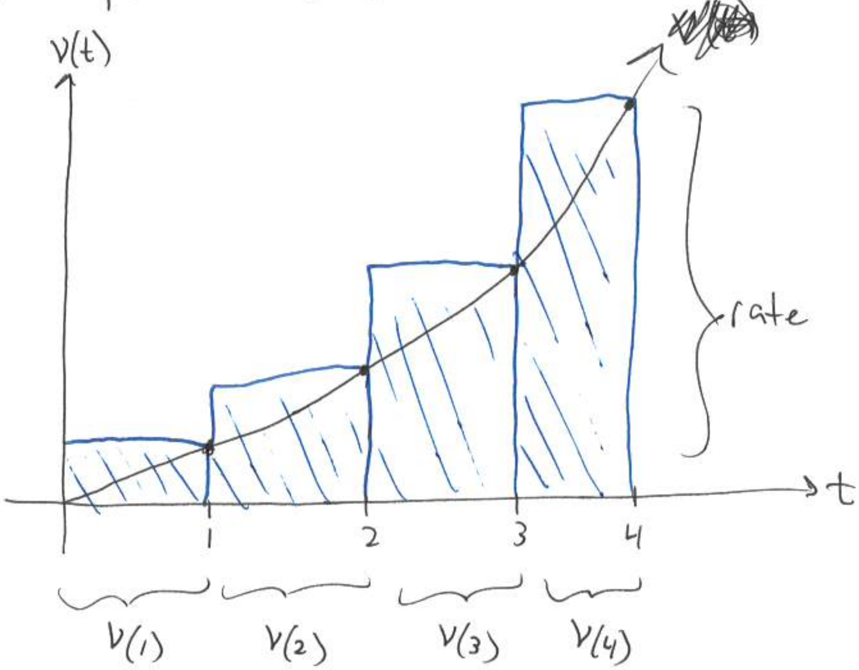
... = 150 miles. ✓

But that wasn't very realistic. What car begins moving at a CONSTANT velocity?! To be more realistic, the velocity should be a function of time.

QUESTION – Suppose the velocity of a car is modelled by the equation $V(t) = t^3 + t$ measured in MPH. How far does the car travel between $t=0$ and $t=4$ hours?

This question is quite a tad harder to answer. We could estimate it, and say something like: "the car is starting at $V(0)=0$ MPH and ends up at $V(4) = 4^3+4 = 68$ MPH, so we could 'estimate' the average speed was something like $V(2) = 2^3+2 = 10$ MPH and say it travelled $10 \times 4 = 40$ miles. Or ~~the~~ since that's way lower than the max speed of 68 MPH, we could guestimate the average speed was $\frac{68}{2} \times 4 = 136$ miles. Those are way different though ☹. Here's a bright idea: lets cut up the time interval into pieces and use this same reasoning on those little pieces. Take the interval $[0,4]$ and cut it into hour intervals $[0,1] \cup \ldots$ $\ldots \cup [1,2] \cup [2,3] \cup [3,4]$, and for the sake of our sanity while making this calculation we'll ~~assume~~ estimate on each interval to be constantly the speed it's going at the end of the interval.

So on [2,3] we'll estimate the car is going $v(3) = 3^3 + 3 = 30$ miles/hr. The speed of the car is increasing the whole time, so this will be an overestimation of the total distance travelled, but we can try to improve this later.



Still using the rate × time formula, this will end up giving us

$$(1^3+1)(1) + (2^3+2)(1) + (3^3+3)(1) + (4^3+4)(1)$$

$$= 2 + 10 + 30 + 68 = 110 \text{ miles}$$

Notice that each of these summands gives us ~~the~~ the area of a rectangle in the graph about, with height giving the rate and width giving the time. Neat! But I want a better estimate for how far the car travelled. What if we use this same technique, but subdivide the time intervals even further? Say into half-hours?

$$[0,4] = [0, \tfrac{1}{2}] \cup [\tfrac{1}{2}, 1] \cup [1, \tfrac{3}{2}] \cup \cdots \cup [\tfrac{7}{2}, 4]$$

In groups of 3? 4? 5?, do this! And I'll float.

ESTIMATE!!!

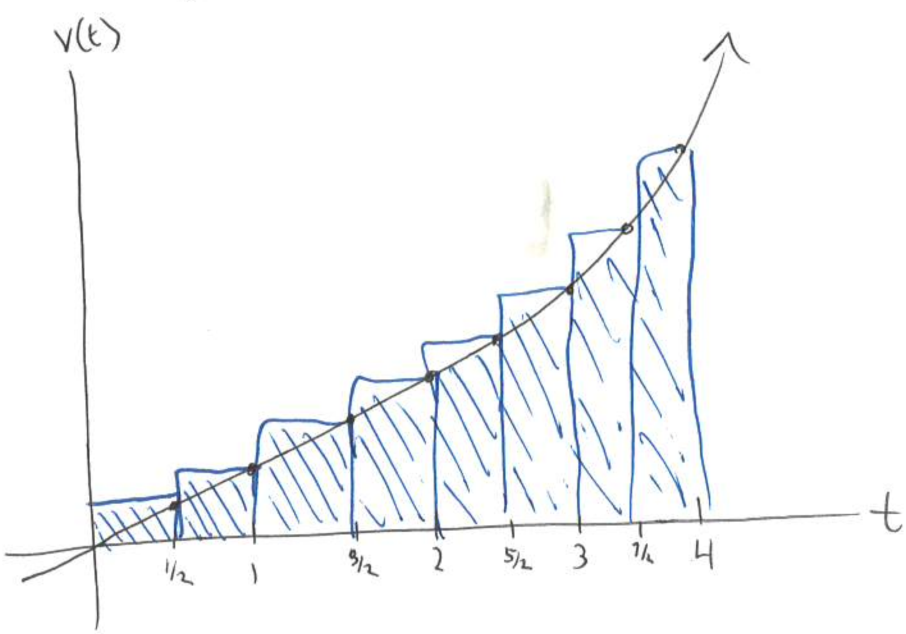
I get $\frac{1}{2}\left(v(\tfrac{1}{2}) + v(1) + \cdots + v(4)\right)$

$$= \frac{1}{2}\left(\frac{1}{2} + 2 + \frac{39}{8} + 10 + \frac{145}{8} + 30 + \frac{361}{8} + 68\right)$$

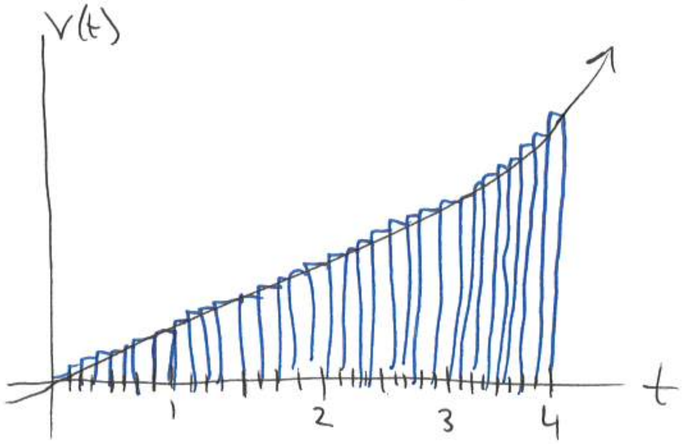
$$\approx \frac{1}{2}\left(1 + 2 + 5 + 10 + 18 + 30 + 45 + 68\right)$$

$$\approx \frac{1}{2}(180) = 90 \text{ MILES}$$

Notice the rectangles in the graph we get for this estimation though:



If we were to keep doing this, we'd keep getting closer to the true distance that the car travelled, but our calculations will get tougher and tougher. In terms of those rectangles, though, they'd get thinner and thinner, eventually giving us a picture that looks like.



In sum, as the rectangles get smaller and smaller, their area will approach the area below the graph of $v(t)$ between $t=0$ and $t=4$. So this AREA 70 is the answer to our question. Our physics question is equivalent to this GEOMETRY question! So then the focus of this week will turn towards this geometry question: what is the LIMIT of these areas as we shrink the widths of these rectangles to zero?

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Office Hours
10am-3pm
@ Skye 284 (Above us)

- What the class is about: integrals (backwards derivatives), which help you solve ODEs. Modelling!!!
- Homework: just a list of problems. Assessments worth points.

THIS Class Will feel hard, but assuredly,

- if I think you'll be fine in any class for which this is a prereq, you'll pass.
- Grading Scheme: (it's going to be okay, you'll almost certainly pass and please please talk to me if your worried) $G = F + \frac{1}{2}A(100-F)$
- Textbook link: file.io/lfLTbC
  file.io/3jfN4G ✓

Neuhauser.

I'll give you chapters if you wanna follow along

Use Youtube & Desmos & Wolfram Alpha.

3. Final: Sat Aug 31 8-10AM in here. (Sorry)
Schedule ahead of time

2. Discussion Wed 11:40-1:30pm @ Skye 171.

- Happiness
- Math is hard

1. Who read the syllabus!?

Read Syllabus!

I put Solutions to some HW.

Tell me if you think one is wrong

Homework is challenging: come to office hours.

feel free to just work in office hours.

Int BioCalc Week 1 Tuesday

WolframAlpha Desmos Youtube

Yesterday we talked about the conceptual underpinnings of the whole course, so, I've gotta ask: y'all have any questions?

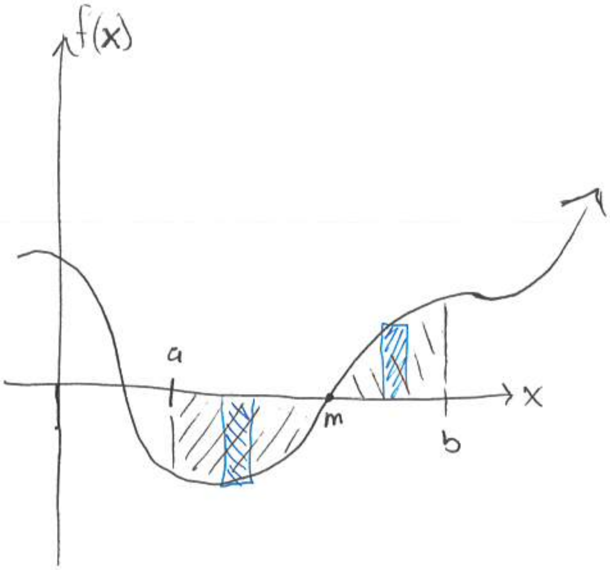
Today we're doing math! The most tractable approach to yesterday's question was the geometric version: what's the area under that curve? Let's state this more generally:

QUESTION – Given a continuous function f, how can we find the area between ~~th~~ the graph of f and the x-axis between two real numbers a and b?

~~Since~~ Lets represent this signed area from a to b by the notation

$$\int_a^b f \qquad \text{OR} \qquad \int_a^b f(x)\,dx$$

This is called the ~~integr~~ definite integral of f ~~between~~ from a ~~and~~ to b. The notation on the right will make more sense later. ~~When~~ Given an integral $\int_a^b f(x)\,dx$, we call $f(x)$ the integrand.



Now we're gotta be careful when we say "area" now, because we're talking about ~~signed area~~; we can have regions that live below the x-axis when the heights of our rectangles would be negative.

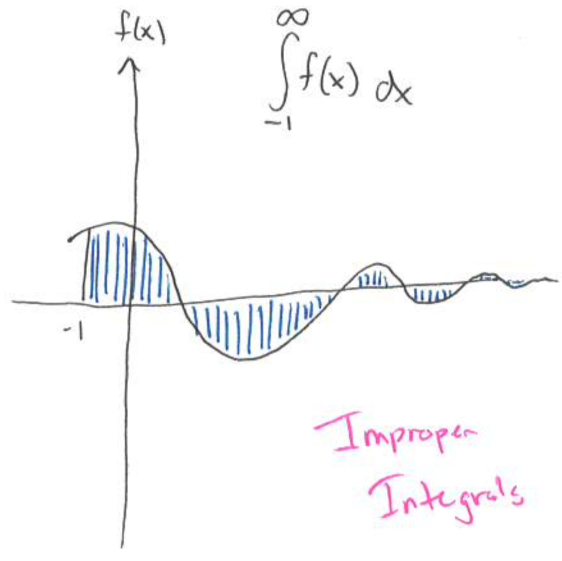
But we also have a sense of direction to our independent variable x. An integral $\int_a^b f$ will have the opposite sign of the integral $\int_b^a f$.

Will each of the following be positive or negative?

$$\int_a^m f(x)\,dx \qquad \int_m^b f(x)\,dx \qquad \int_m^a f(x)\,dx \qquad \int_b^m f(x)\,dx$$

And although we ~~work much~~ won't work with them until later in the course, the bounds on an intergral certainly might be infinite.

$$\int_{-1}^{\infty} f(x)\,dx$$



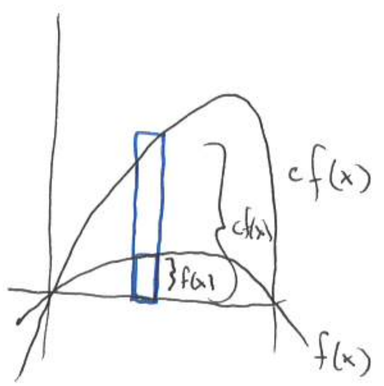
Improper Integrals

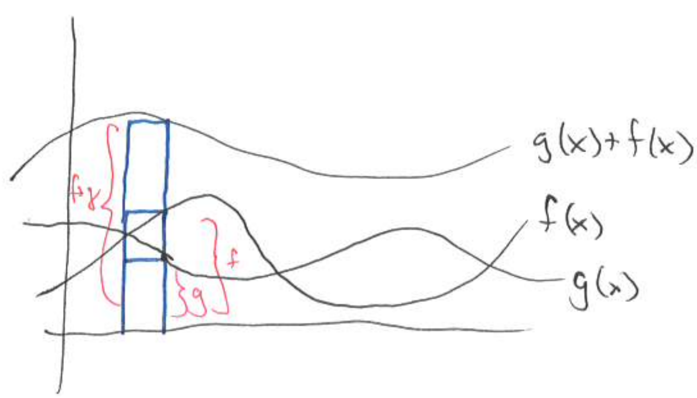
And furthermore, this $\int$ integral operation will be linear. This means that you can pull out constants and break apart integrals across sums.

$$\int_a^b c f(x)\, dx = c \int_a^b f(x)\, dx$$

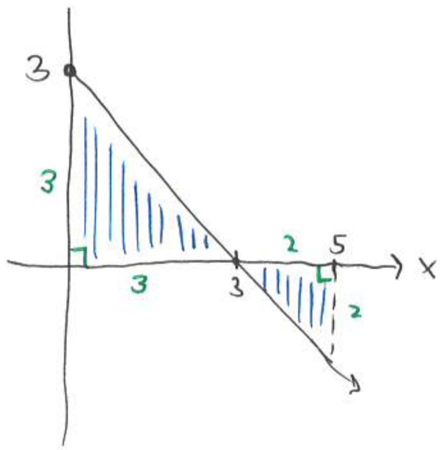
$$\int_a^b f(x) + g(x)\, dx = \int_a^b f(x)\, dx + \int_a^b g(x)\, dx$$

And you can see this works because the integrand is really just the height of all the rectangles we'd use to appoximate an area





Certain integrals will be pretty easy to evaluate just based on the shape of their graphs and some geometry. For example $\int_0^5 (-x+3)\,dx$. Looking at



the graph of this function, this integral will just be the signed area of those two triangles

$$\int_0^5 (-x+3)\,dx = \frac{1}{2}(3)(3) - \frac{1}{2}(2)(2)$$
$$= \frac{1}{2}(9-4) = 5/2$$

So general advice: drawing a graph of your function can be super helpful.

In small groups, try these ones

$$\int_4^7 2x-3\,dx \qquad \int_1^4 \lfloor t+2 \rfloor\,dt$$

$\lfloor 1/2 \rfloor = 0$

$\lfloor 1 \rfloor = 1$

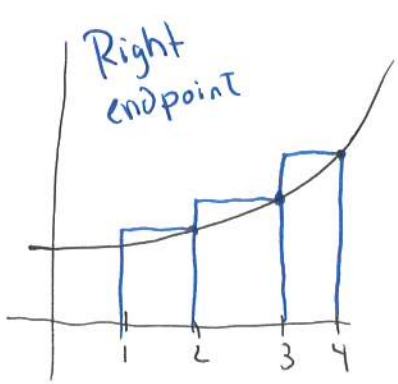
$\lfloor 2.3 \rfloor = 2$

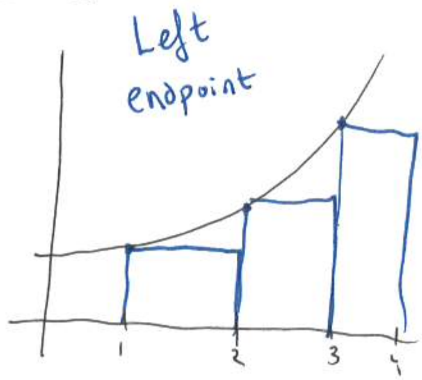
$\lfloor \pi \rfloor = 3$

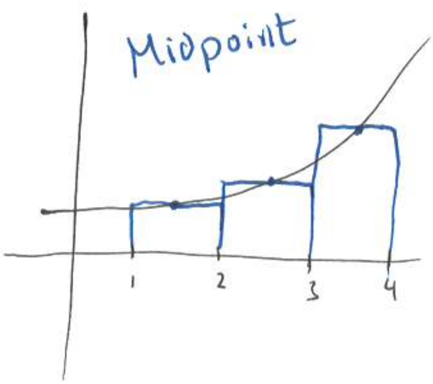
$\lfloor -7.6 \rfloor = -8$

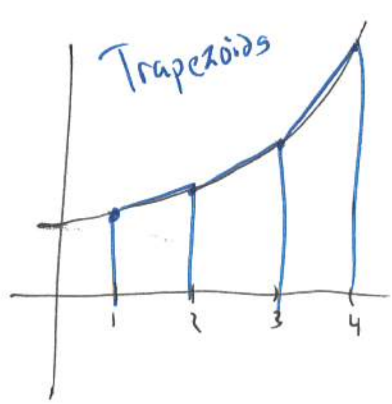
$$\int_{-2}^{2} |2-2x|\,dx$$

For some functions though, we can't find their integrals using simple geometry, so we're gotta go back to the idea of approximating rectangles again. Let's look at $\int_1^4 x^2+1 \, dx$. Recall that last time we did this thing with rectangles we used the right-hand side of our little subintervals as the height of our rectangles. But this was an ~~arbitry~~ arbitrary choice. We could have used the left endpoint, or even the midpoints, or even used trapezoids instead of rectangles :. All of these ideas are methods of approximating an integral using <u>Riemann Sums</u>.









• Approximate $\int_1^4 x^2+1 \, dx$ by dividing the domain of integration into 3 subintervals and using a left-endpoint Riemann sum. Approximate it with 2 subintervals using a right-endpoint Riemann sum.

Write down a midpoint Riemann sum approximating $\int_1^4 x^2+1 \, dx$ by subdividing the domain of integration into 6 subintervals.

I hinted last lecture that we could formally define the integral as ~~the~~ a LIMIT of these Riemann sums as we let the width of the rectangles approach 0. Let's do this. Writing this will also explain the notation

$$\int_a^b f(x) \, dx .$$

Suppose we want the area under the curve of f between a and b. Let's divide this domain of integration into n subintervals. So the width of each subinterval will be $\frac{b-a}{n}$, and we will be

Summing up the areas of n rectangles:

$$\sum_{i=1}^{n} (\text{height}) \times (\text{width})$$

RIGHT-ENDPOINT

$$= \sum_{i=1}^{n} f\left(a + i\frac{b-a}{n}\right) \cdot \frac{b-a}{n}$$

$$= \left(\sum_{i=1}^{n} f\left(a + i\frac{b-a}{n}\right)\right) \frac{b-a}{n}$$

$\uparrow$ ∂x or Δx

Then we simply take a limit and define this to be our integral of f.

$$\int_a^b f(x)\, \partial x := \lim_{n\to\infty} \sum_{i=1}^{n} f\left(a + i\frac{b-a}{n}\right) \frac{b-a}{n} \quad (*)$$

So $\int$ is like a "continuous summation."

(How would (*) change if we used left-endpoints instead?)

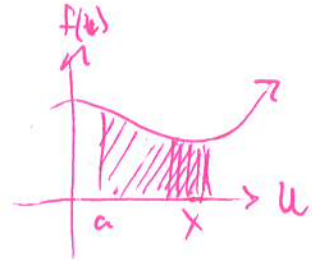
Int Bio Calc Week 1 Wednesday

The Fundamental Theorem of Calculus

Warning: Math
Don't take notes?
Except for theorems and defs.

Today we'll flush out the relationship between integrals and derivatives. The ~~first~~ inspired bit is to let one of our bounds of an integral be variable. Define the function

$$F(x) = \int_a^x f(u)\, du$$



QUESTION* – What is the derivative of F?

$$\frac{\partial}{\partial x} F(x) = \frac{\partial}{\partial x} \int_a^x f(u)\, du$$

$$= \lim_{h \to 0} \left(\int_a^{x+h} f(u)\, du - \int_a^x f(u)\, du \right) \frac{1}{h}$$

$$= \lim_{h \to 0} \frac{1}{h} \left(\int_x^{x+h} f(u)\, du \right)$$

$$= \dots$$

$$\text{intuition} = \lim_{h\to 0} \frac{1}{h}\left(f(x)h\right) = f(x)$$

So $\frac{\partial}{\partial x} F = f$! The derivative operator $\frac{\partial}{\partial x}$ undoes the integral $\int_a^x - \,dt$!

THEOREM (FTC 1) — If f is continuous on $[a,b]$, then the function $F(x) = \int_a^x f(u)\,du$ will be differentiable on (a,b), with derivative $f(x)$. $\frac{\partial}{\partial x}\int_a^x f(u)\,du = f(x)$ (*)

Neat! But what about the other way? Does the integral operator "undo" the differential $\frac{\partial}{\partial x}$ too?
At least a little bit. (*) Tells us that F is <u>an antiderivative</u> of f, that is a function with a derivative of f. But there a plenty of these though. If $\frac{d}{dx}F^{4}(x) = f(x)$ Then $\frac{\partial}{\partial x}\left(F(x) + C\right) = f(x)$ too, for ANY real number C.
So we ~~can compute the~~ have an inverse "up to some constant C." This isn't useless. We should invent some term for this.

Define the indefinite integral of f, denoted

$\int f(x)\,dx$ (without bounds) as the family of antiderivatives

of f, presented as $F(x) + C$, for some real number C.

But how do we evaluate definite! integrals! Like

if we have $\int_a^b f(x)\,dx$, this ~~box~~ is an area an NUMBER,

so how can we ditch the +C?

Well, suppose F is some antiderivative of f, but so

is G. So

$$G(x) = \int_a^x f(u)\,du \quad (**)$$

G differs from F only by a constant, so

$$G(x) = F(x) + C \quad (\ddagger)$$

But $G(a) = 0$, so $F(a) = -C$, and $(\ddagger)$ becomes

$$G(x) = F(x) - F(a)$$

Plugging this into $(**)$ we get

$$G(x) = \int_a^x f(u)\,du = F(x) - F(a)$$

Finally evaluating this at b we get

$$\int_a^b f(u)\,du = F(b) - F(a) \checkmark$$

(FTC 2)

THEOREM — If f is continuous on (a,b), then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

when $\frac{d}{dx}F(x) = f(x)$.

We did it! How does everyone feel?

That was a lot, right? Let's do some commonly asked exercises now that utilize the FTC.

Take ~~the~~ the following derivatives (Clean, Brand)

$$\frac{d}{dx}\int_0^{x^2}(u^3-2)\,du.$$

$$\|$$

$$\frac{d}{dx}F(x^2) = f(x^2)\frac{d}{dx}(x^2) \qquad ((x^2)^3-2)(2x)$$

$$2x^{\cancel{6}7} - 4x$$

f	$\int f$
x^n	$\frac{1}{n+1} x^{n+1} + C$
e^x	$e^x + C$
$\sin(x)$	$-\cos(x) + C$
$\cos(x)$	$\sin(x) + C$
$\sec^2(x)$	$\tan(x) + C$
$\sec(x)\tan(x)$	$\sec(x) + C$
$\frac{1}{x^2+1}$	$\arctan(x) + C$
$1/x$	$\ln(x) + C$
b^x	$\frac{b^x}{\ln(b)} + C$
$\sin(x)$	$-\cos(x) + C$
$\csc^2(x)$	$-\cot(x) + C$
$\csc(x)\cot(x)$	$-\csc(x) + C$

$\int \tan(x)\,dx$? later...

$\int \ln(x)$? later

Do first

Evaluate $\int e^x + \sin(x)\, \partial x$.

$e^x - \cos(x) + C$

Evaluate $\int x^5 \,\partial x$

power rule backwards

"check your answer!"

$\frac{1}{6}x^6 + C$

Evaluate $\int_{-\pi/2}^{\pi/3} 2\cos(\theta)\, \partial\theta$

$= 2\int \dots$

$= 2\left(\sin\theta\right)\Big|_{\theta=-\pi/2}^{\theta=\pi/3}$

$= 2\left(\sin\left(\frac{\pi}{3}\right) - \sin\left(-\frac{\pi}{2}\right)\right) = 2\left(\frac{\sqrt{3}}{2} + 1\right)$

$= \sqrt{3} + 2$

$\int_0^{\pi/3} \sec^2(t)\, \partial t$

$\int x^2 + 3x \, \partial x$

$\int_{-2}^{2} 2x^2 - 3x^3 \, \partial x$

$\frac{\partial}{\partial x}\int_{x^2}^{x^3} e^u \, \partial u$

style...

$\int \frac{1}{t} + \frac{1}{t^2}\, \partial t$

$\int_2^{\pi} \frac{7}{\mu}\, \partial\mu = \int \frac{7\, d\mu}{\mu}$

$\frac{\partial}{\partial t}\int_{\ln(t)}^{\pi} \sin^2(x)\, \partial x$

$\int \sec\theta \tan(\theta)\, \partial\theta$

Int Bio Calc Week 1 Thursday

We didn't have much time yesterday to practice doing definite/indefinite integrals. So today, let's start just by practicing those a bunch.

Evaluate

$$\frac{\partial}{\partial x}\int_{x^2}^{x^3} e^u \, du$$

$$\int x^2 + 3x \, dx$$

$$\int \frac{1}{t} + \frac{1}{t^2} \, \partial t$$

"Check your anti-derivative by taking the derivative!"

$$\frac{\partial}{\partial t}\int_{\ln(t)}^{\pi} \sec^2(t) \, \partial t$$

$$\int \sec(\theta)\tan(\theta) \, \partial\theta$$

$$\int_0^{\pi/3} \sec^2(t) \, \partial t$$

$$\int_{-2}^{2} 2x^2 - 3x^3 \, dx$$

$$\int_2^{\pi} \frac{7 \, \partial\mu}{\mu}.$$

Here are a few "sample solutions" I can throw up on the board before class.

$$\frac{\partial}{\partial x}\int_{x}^{\sin(x)} \cos(u^2)\, du = \frac{\partial}{\partial x}\int_{x}^{0} \cos(u^2)\, du + \frac{\partial}{\partial x}\int_{0}^{\sin(x)} \cos(u^2)\, du$$

$$\cdots = -\cos\left(x^2\right) + \cos\left(\sin^2(x)\right)\left(\cos(x)\right) \mathbin{/\!/}$$

$$\int_{\pi}^{e} 2x^5 - 7\sin(x)\, dx = 2\int_{\pi}^{e} x^5\, dx - 7\int_{\pi}^{e} \sin(x)\, dx$$

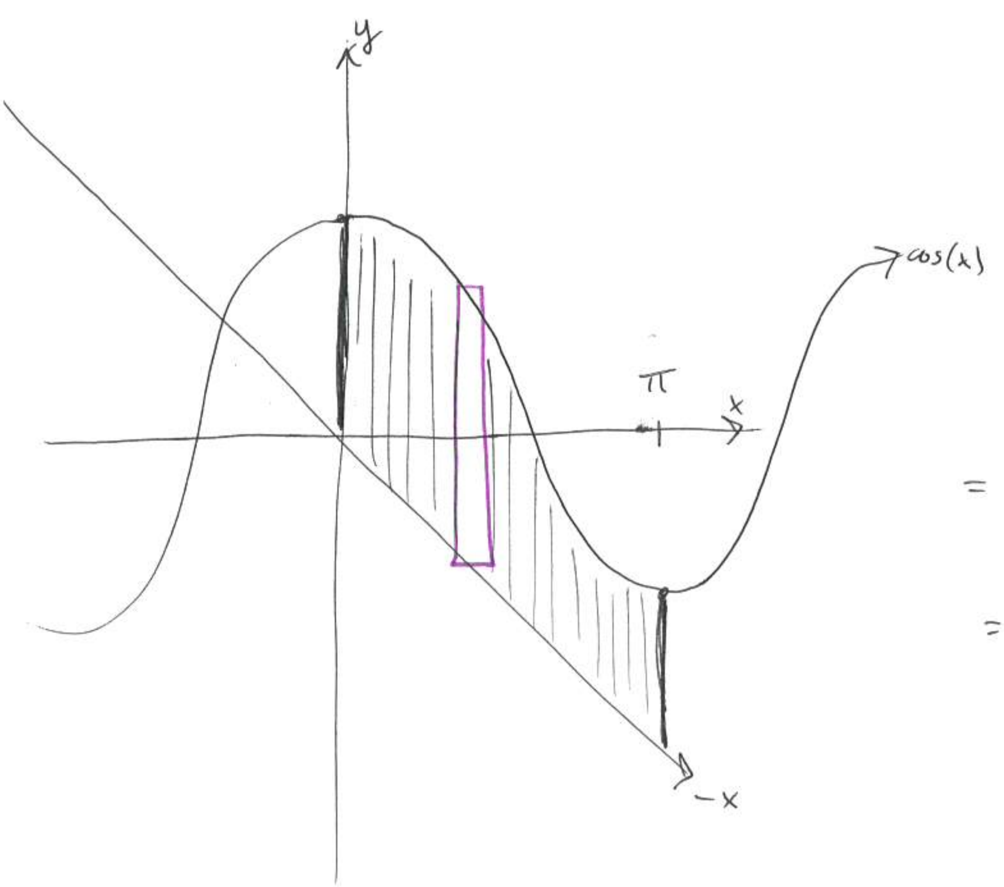
$$\cdots = 2\left(\tfrac{1}{6}x^6\right)\Big|_{\pi}^{e} - 7\left(-\cos(x)\right)\Big|_{\pi}^{e}$$

$$\cdots = \frac{1}{3}\left(e^6 - \pi^6\right) + 7\left(\cos(e) + 1\right) \mathbin{/\!/}$$

$$\int \tfrac{1}{2}x + 2^x\, dx = \tfrac{1}{4}x^2 + \frac{1}{\ln(2)}2^x + C$$

We could get a little more fancy and use our mastery of integrals to tackle some tougher geometry questions:

QUESTION – Find the (unsigned) area of the region(s) in the plane bound by the curves $y = \cos(x)$, $y = -x$, $x = 0$, and $x = \pi$.



$$\int_0^{\pi} \text{"top curve"} - \text{"bottom curve"} \, dx$$

$$= \int_0^{\pi} \cos(x) - (-x) \, dx$$

$$= \sin(x)\Big|_0^{\pi} + \tfrac{1}{2}x^2\Big|_0^{\pi} \, dx$$

$$= (0 - 0) + \tfrac{1}{2}(\pi^2)$$

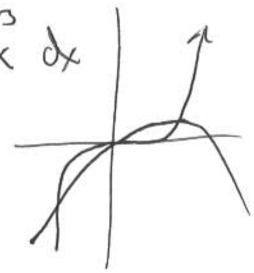
$$= \tfrac{1}{2}\pi^2 \;/\!/$$

Set up the integral(s)

QUESTION – What's the (un-signed) area ~~box~~ of the region in the (x,y)-plane bounded by the curves

$$y = x^3 \quad \text{and} \quad y = -x^2 + 6x$$
$$= x(6-x)$$

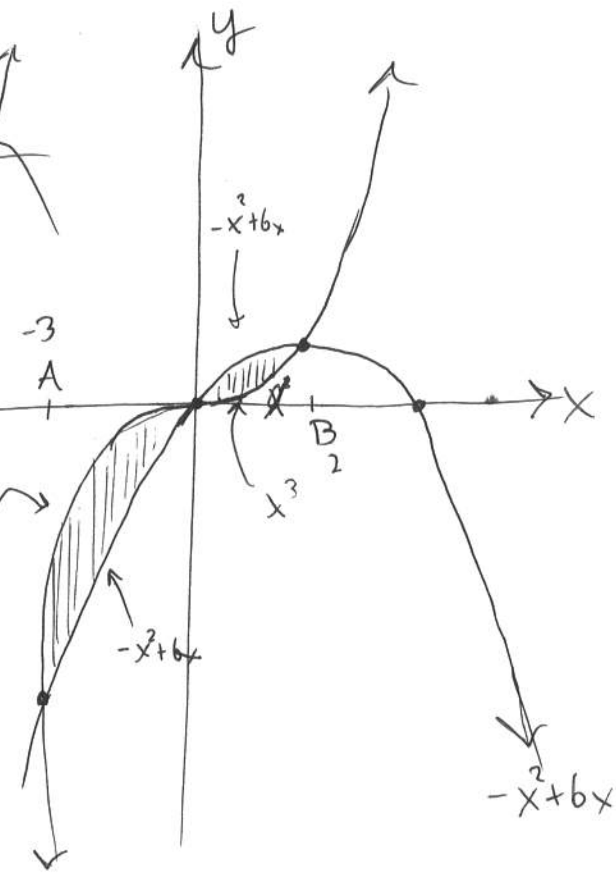
$$\int_{\cancel{A}\,-3}^{0} x^3 - (-x^2+6x)\,dx + \int_{0}^{\cancel{B}\,2} (-x^2+6x) - x^3\,dx$$



$$\cancel{\int_{-3}^{0} -x^3 - x^2 + 6x\,dx + \int_{0}^{2}}$$

$$= \int_{-3}^{0} x^3 + x^2 - 6x\,dx + \int_{0}^{2} -x^3 - x^2 + 6x$$

STOP HERE...



$$= \frac{1}{4}x^4 + \frac{1}{3}x^3 - 3x^2 \Big|_{-3}^{0} - \left(\frac{1}{4}x^4 + \frac{1}{3}x^3 - 3x^2\right)\Big|_{0}^{2}$$

$$x^3 = -x^2 + 6x$$

$$= 0 - \left(\frac{1}{4}(-3)^4 + \frac{1}{3}(-3)^3 - 3(-3)^2\right) - \left(\frac{1}{4}(2)^4 + \frac{1}{3}2^3 - 3(2)^2\right) - 0$$

$$\Rightarrow x^3 + x^2 - 6x = 0$$

$$\Rightarrow x(x+3)(x-2) = 0$$

$$= -\frac{81}{4} + \frac{27}{3} + 27 - \frac{4^2}{4} - \frac{8}{3} + 12$$

= Calculation...

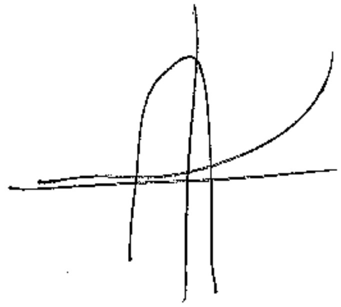
The curvs

$y = e^x$

$y = -x^2 + 7$

~~y~~ $x = \cancel{8} -1$

$x = 1$



curves

$y = \tan(x)$ ✓

$y = 1 \quad x = -1$