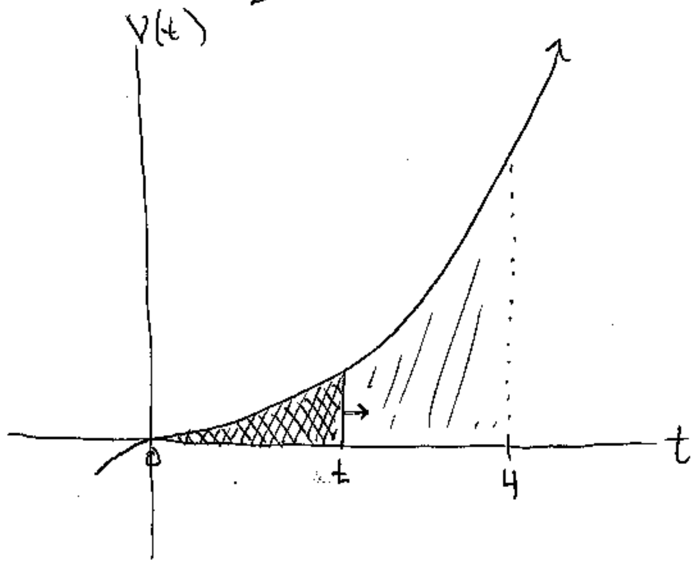
IntBio Calc Week 2 Monday

Let's revisit that very first question we looked at

QUESTION – Suppose the velocity of a car is modelled by the equation $v(t) = t^3 + t$ measured in miles/hr. How far does the car travel between $t=0$ and $t=4$ hours?

We now have the tools to answer this question precisely.



As t moves from 0 to 4, we can think of the car as "accumulating" distance, and geometrically, this distance corresponds to the area under the graph of $v(t)$ that is being accumulated as t goes from 0 to 4. So the total distance travelled is $\int_0^4 t^3 + t \, dt = \frac{(4)^4}{4} + \frac{(4)^2}{2} = 72$ miles.

But we can totally do more than that now too!

QUESTION - Suppose that car is on a straight Stretch of freeway headed towards or away from you. How can you write a function of time that models how far the car is from ~~your~~ you?
(add later: given it starts 20 miles from you, headed away from you?)

So given our velocity function, we need to get a position function from it, but this must be just an antiderivative, an indefinite integral, of $v(t)$!

$$p(t) = \int v(t)\,dt = \int t^3 + t\,dt = \frac{1}{4}t^4 + \frac{1}{2}t^2 + C$$

So $p(t) = \frac{1}{4}t^4 + \frac{1}{2}t^2 + C$. But note that this $+C$ makes sense. We start with information about the velocity of the car, but NOTHING about the position.

The car ~~[illegible]~~ could have velocity ~~[illegible]~~ $v(t) = t^3 + t$ right next to you on the freeway, or hundreds of miles away from you on the freeway. To get rid of that $+C$, and get a precise function for the car's distance from you, we need some information about the position of the car. Just one nugget of information about its initial position or future position so that we can anchor it to some place on the freeway at some time, and let $v(t)$ dictate the position of the car thereafter.

|| Addendum: the car starts 20 miles from you, headed away.

This tells us that $p(0) = 20$. So we can solve for C!

$$20 = p(0) = \frac{1}{4}(0)^4 + \frac{1}{2}(0)^2 + C \Rightarrow C = 20$$

So $p(t) = \frac{1}{4}t^4 + \frac{1}{2}t^2 + 20$.

This is a tiny example of what's called an initial value problem. We'll talk about more advanced initial value problems in week 5. Let's do one more, less tiny, example.

QUESTION – Suppose you have a ~~population~~ population of bacteria in a petri dish ~~in a petri dish~~ whose ~~population~~ size can be expressed as a function of time $P(t)$ after t days. Suppose we know the population dynamics of the bacteria is given by

$$P'(t) \quad \frac{dP}{dt} = 3000 e^{-t}$$

$$3000 \approx e^{8}$$

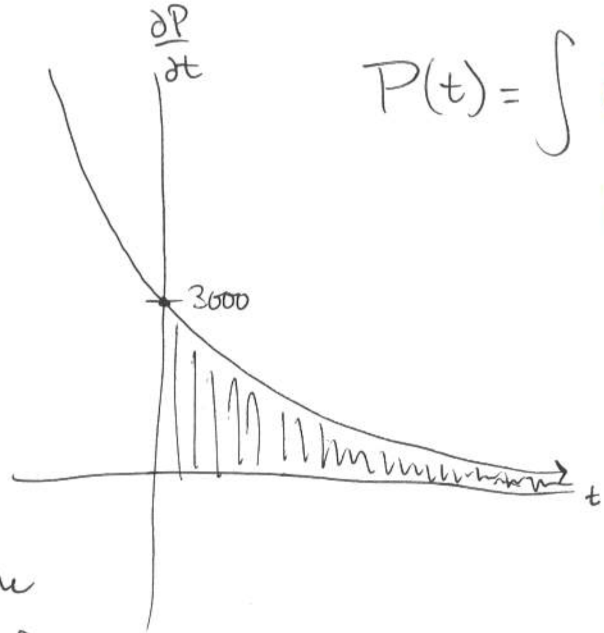
~~If you need a population of bacteria in your petri dish in a week how big of a population do you need to start with?~~

If we know that the population maxes out at 5000 members in the petri dish, how many initial members must there have been?

Based on the graph of $P'(t)$ $\frac{dP}{dt}$, how can we guess that the population might have an upper constraint on its size?

Let's graph

$P'(t)\ \cancel{\frac{dP}{dt}} = 3000e^{-t}$. As t approaches ∞, the area under the curve narrows, so the population is accumulating fewer and fewer members.



$P(t) = \int \frac{\partial P}{\partial t}\, \partial t$

$P'(t)$

So it's reasonable that, if it narrows "quickly enough", then the population maxes out.

Now really $P(t) = \int P'(t)\,\partial t = -3000e^{-t} + C$. If we know that the population maxes out at 5000, this means

$$\lim_{t\to\infty} P(t) = 5000$$

$$\iff \lim_{t\to\infty}\left(-3000e^{-t} + c\right) = 5000$$

$$\Rightarrow C = 5000$$

$$\Leftarrow C = 5000$$

So $P(t) = 5000 - 3000e^{-t}$, and initially at time zero we have

$$P(0) = 5000 - 3000e^{-(0)} = 5000 - 3000$$
$$= 2000 \text{ bacteria.}$$

Break Time

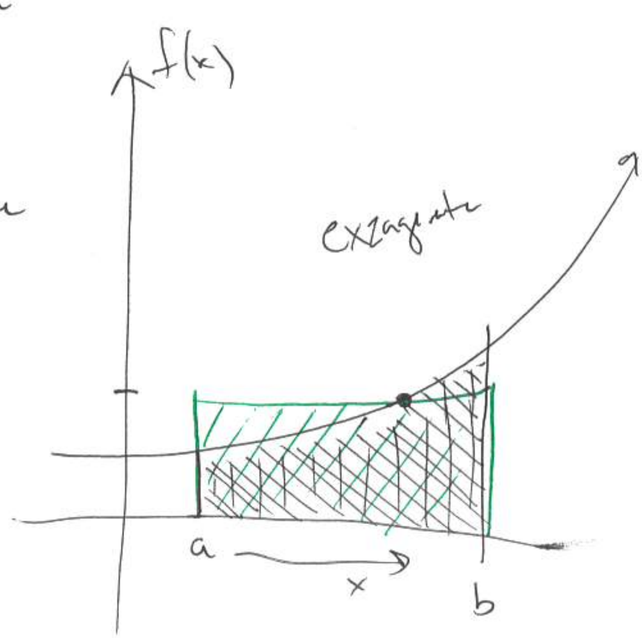
QUESTION – How can we find the average value of a function on an interval?

I'll tell you:

If f is continuous on $[a,b]$, then its average value on $[a,b]$ is given by $\frac{1}{b-a}\int_a^b f(x)\,dx$.

Now let me tell you why:

Again think of f as "accumulating" something as x sweeps from a to b, and $\int_a^b f(x)\,dx$ counts how much was accumulated. If we replace $f(x)$ in $\int_a^b f(x)\,dx$ with the average value of f, say A, then it should sweep out the same area. So we should have



$$\int_a^b A\,dx = \int_a^b f(x)\,dx$$

$$\Rightarrow \quad Ax\Big|_a^b = \int_a^b f(x)\,dx$$

$$\Rightarrow \quad A(b-a) = \;''$$

$$\Rightarrow \quad A = \frac{1}{b-a}\int \;''$$

And from this ~~we get~~ we get a nice theorem

Mean Value Theorem for Definite Integrals – If $f(x)$ is continuous on $[a,b]$, then there exists some $c \in [a,b]$ that achieves the average value of f. So

$$f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

A particle moves along the x-axis with velocity

$$v(t) = -(t-3)^2+5$$

for $t \in [0,6]$.

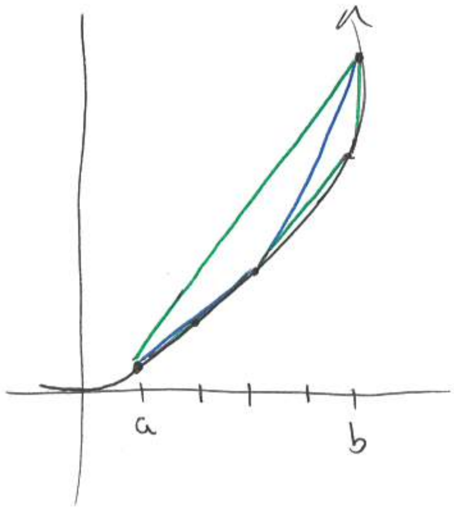
- Graph $v(t)$.
- Find the average velocity of the particle on the time interval $[0,6]$.
- Find $\bar{t} \in [0,6]$ such that the velocity of the particle at $\bar{t}$ is the same as the average velocity. Does more than one possible $\bar{t}$ exist? How could ~~the~~ you find $\bar{t}$ graphically?

Int Bio Calc Week 2 Tuesday

Today we'll talk about how to calculate the arclength of a curve.

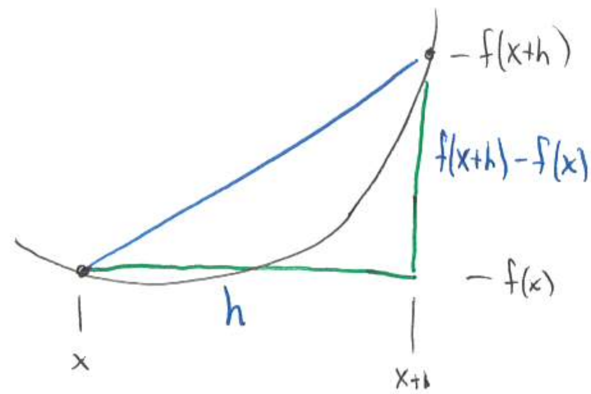
QUESTION – How do you calculate the length of the curve $y = x^2$ between the points $\overset{(0,0)}{\cancel{(a,a^2)}}$ and $\overset{(3,9)}{\cancel{(b,b^2)}}$?

The idea ~~between~~ behind the procedure is a useful one.



- Take your interval (a,b) and subdivide it into n subintervals, say of length h $\left(= \frac{b-a}{n}\right)$.
- Take each subinterval $(x, x+h)$ and find the length of the straight line connecting $(x, f(x))$ to $(x+h, f(x+h))$.
- Add up all these lengths in a general formula, and take the limit as n, the number of subintervals, goes to ∞ (or equivalently as $h \to 0$).

First, how do we find the length of one of those straight lines between $(x, f(x))$ and $(x+h, f(x+h))$. Pythagorean Theorem! It's



$$\sqrt{h^2 + (f(x+h) - f(x))^2}.$$

Now to sum these lengths all up for each subinterval (we're going to get a little fuzzy here); and take the limit as h tends to zero (the width of each subinterval)

$$\lim_{h \to 0} \sum_{\substack{\text{over all} \\ \text{subintervals} \\ \text{from } a \text{ to } b}} \sqrt{h^2 + (f(x+h) - f(x))^2}$$

$$= \lim_{h \to 0} \sum_{\substack{\text{over all} \\ \text{subintervals} \\ a \text{ to } b}} \sqrt{h^2 + h^2 \left(\frac{f(x+h) - f(x)}{h} \right)^2}$$

$$= \lim_{h \to 0} \sum_{''} \sqrt{1 + \left(\frac{f(x+h) - f(x)}{h} \right)^2} \; h$$

width

$$= \int_a^b \sqrt{1 + (f'(x))^2} \, dx$$

This is our formula for arclength!

So to answer that particular question, the arclength of $y = x^2$ between $(0,0)$ and $(3,9)$ is

$$\int_0^3 \sqrt{1 + (f'(x))^2}\, dx$$

$$= \int_0^3 \sqrt{1 + \underbrace{(2x)^2}_{4x^2}}\, dx$$

This is a tough integral to evaluate because of that square root, ... but I've got this, via Wolfram Alpha

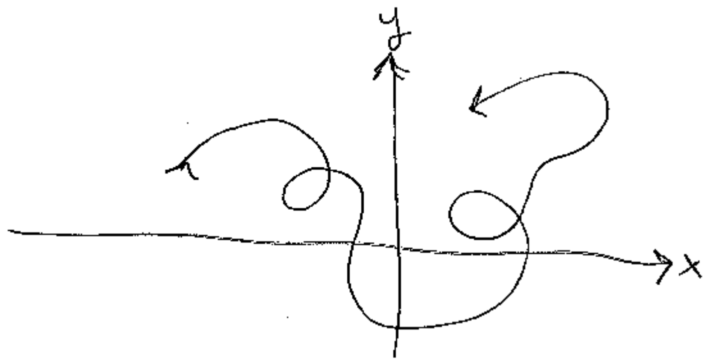
$$\frac{1}{4}\left(6\sqrt{37} + \sinh^{-1}(6)\right) \approx 9.7471$$

arcsinh
"hyperbolic sine"

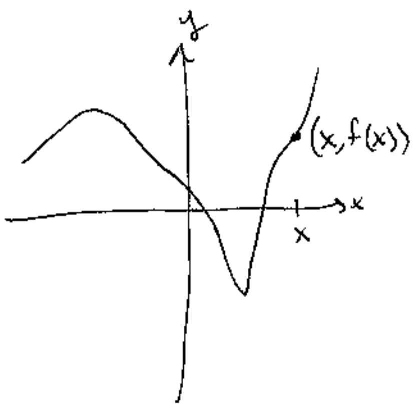
You practice on this do-able one.

Find the length of the quarter-circle given by $y = \sqrt{1-x^2}$ for $x \in [0,1]$. It should be $\frac{1}{2}(2\pi(r)) = \pi$, right!?!

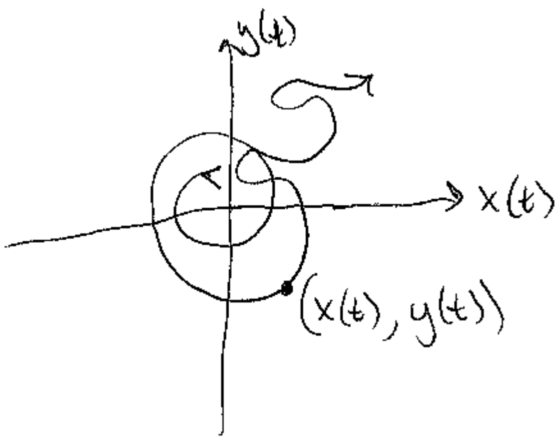
But notice these curves are all given by functions, the graphs of Functions! What about cooler curves, like



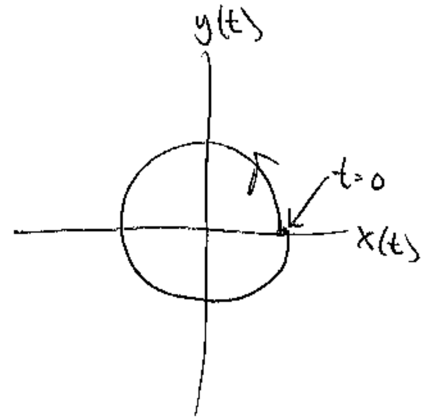
We can't write that curve as the graph of a function, but we certainly can express it ~~at~~ as a <u>parameterized curve</u>. On a graph, your coordinates



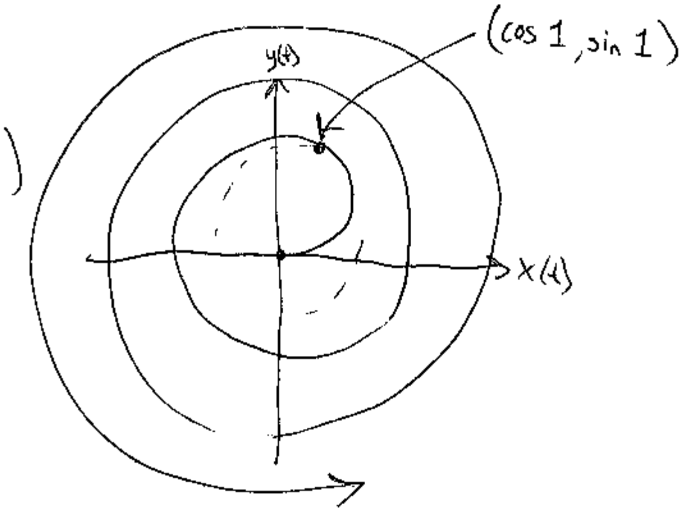
are given as a pair, where one is just your variable and only the $y=f(x)$ coordinate gets to have any fun. What if you represent ~~to~~ each coordinate of a point with its own function $x(t)$ and $y(t)$ of a new variable, a <u>parameter</u> t?



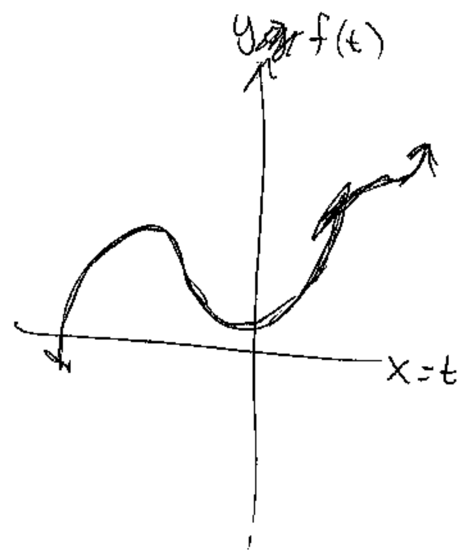
EXAMPLE – The parametric curve $\rho(t) = (\cos(t), \sin(t))$ is a circle in the (x,y)-plane.



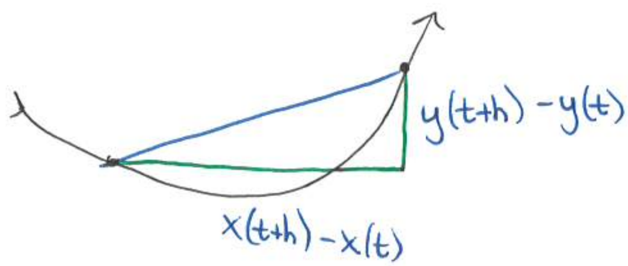
EXAMPLE – The curve given by $\rho(t) = (t\cos t + t\sin(t))$ is a nice spiral.



EXAMPLE – The graph of f is given by the parametric curve $\rho(t) = (t, f(t))$



To find the arclength of a parametrically defined curve, we'll get a more symmetric formula.



$$\lim_{h \to 0} \sum \sqrt{(x(t+h)-x(t))^2 + (y(t+h)-y(t))^2}$$

$$= \lim_{h \to 0} \sum \sqrt{\left(\frac{x(t+h)-x(t)}{h}\right)^2 + \left(\frac{y(t+h)-y(t)}{h}\right)^2}\ h$$

$$= \int_{t=a}^{t=b} \sqrt{(x'(t))^2 + (y'(t))^2}\ dt$$

$$\int_a^b \sqrt{1^2 + (f'(t))^2}\ dt$$

Notice that this becomes our usual formula for

~~x(t)~~ $x(t) = t$

$y(t) = f(t)$

Find the length of the curve

$$y^2 = x^3$$

from $x = 1$ to $x = 4$

As a parametric curve we can write this as

$\rho(t) = (x(t), y(t)) = (t^2, t^3)$, since

$$x(t) = t^2, \quad y = t^3$$

$$\Rightarrow x^3 = t^6 = y^2 =$$

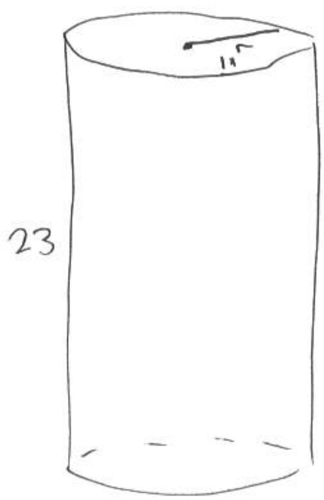
So by our formula, the length of this curve is

$$\int_1^4 \sqrt{2t + 3t^2} \, \partial t \quad //$$

IntBioCalc Week 2 Wednesday

Another neat application of integration is that we can now ~~provid~~ find the volume of irregular ~~shapes~~ solids, provided only some nice information about the cross-sections of the solid along some axis.

You might remember this geometric fact: the volume of a prism, a (compact) solid with every cross-section along some axis congruent, is given by the area of that cross-section times the length of the prism.



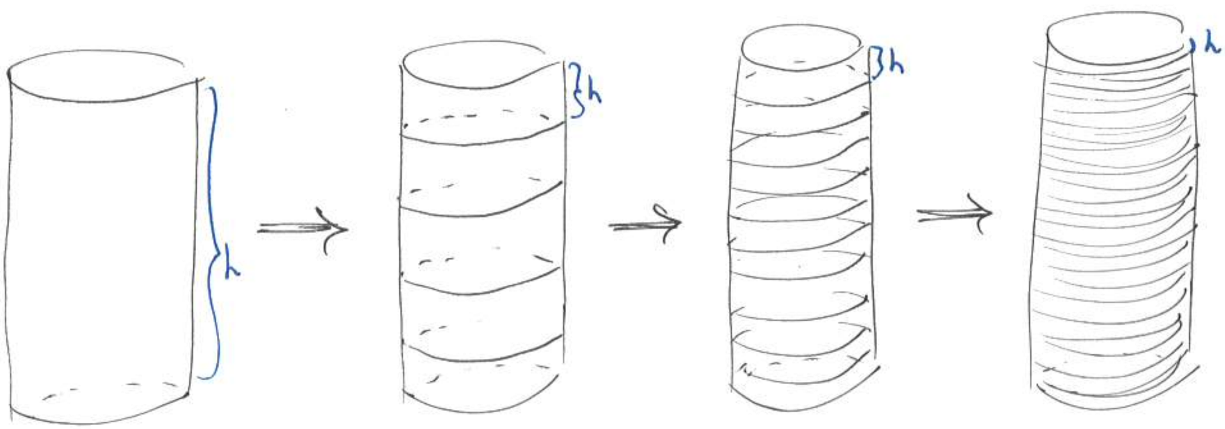
What's the volume of a cylinder with height 23 and a circular base with radius 11?

$$\text{Volume } \cancel{\text{AREA}} = \begin{pmatrix}\text{Cross-section}\\ \text{area}\end{pmatrix} \times (\text{height})$$

$$= \left(\pi (11)^2\right) \times (23)$$

$$= 2783\pi$$

A way to think about this calculation is that we're cutting up the cylinder into a bunch of shorter cylinders (disks) that we're each finding the ~~vol~~ volume of before adding them all up.



Each individual disk in that last picture has a very tiny height h. It's like we're looking at the limit as $h \to 0$ (or as $n \to \infty$ where n is the number of subdisks).

So this calculation has an integral hiding in it.

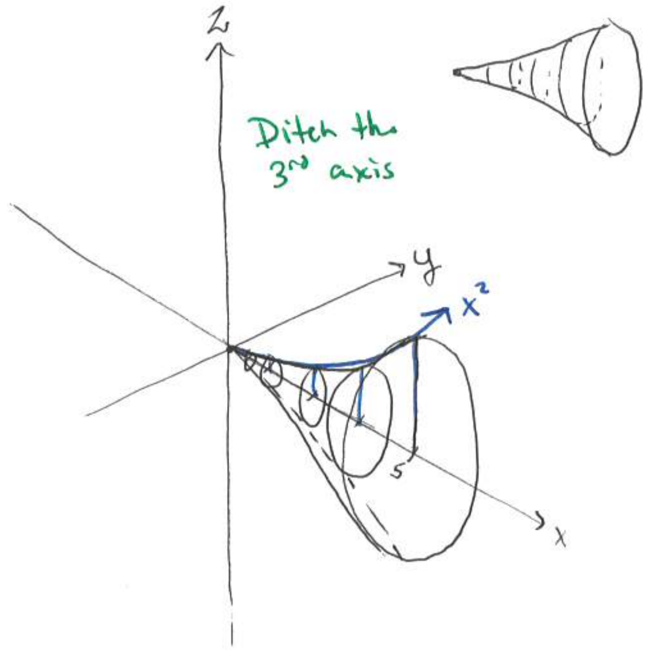
Letting A be the cross-sectional area of the cylinder, we have

$$\int_0^{23} A \, dx = \overset{\pi(11)^2}{\overset{\downarrow}{A}} x \Big|_0^{23} = \pi(11)^2(23-0) = 2783\pi$$

$$= \begin{pmatrix}\text{cross-section} \\ \text{area}\end{pmatrix}(\text{height})$$

In that example, the integrand A was just a constant function, but we ~~can now~~ could let A be any function (of the height) now. So we can now find the volume of solids with varying cross-sectional areas (shapes).

Find the volume of the solid with ~~a cros~~ circular cross-sections that live allong the x-axis between $x=0$ and $x=5$, such that the radius of the circular-cross-section is given by $r=x^2$.



The area of a cross-section is now a function of x, and this function is $A(x) = \pi r^2 = \pi (x^2)^2 = \pi x^4$.

So the volume is:

$$\int_0^5 A(x)\,dx = \int_0^5 \pi x^4\,dx = \pi\left(\frac{1}{5}x^5\Big|_0^5\right)$$

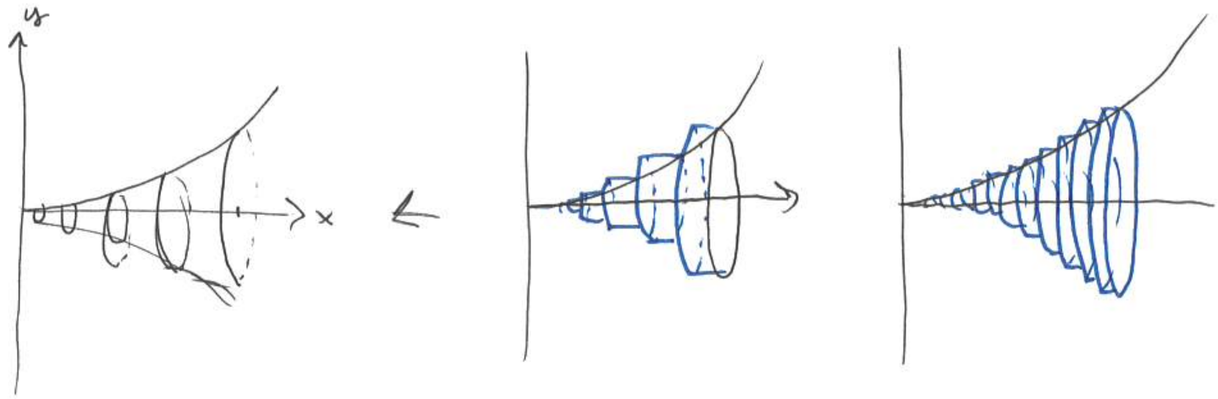
$$= \frac{\pi}{5}5^5 = 625\pi$$

Questions?

I think its important to see that we could have phrased that question, the geometric description of that solid, differently:

> Find the volume of the solid generated by taking the region under the graph of $f(x) = x^2$ ~~and rotating~~ between $x=0$ and $x=5$ and rotating it about the x-axis.

Because of this description, we sometimes call such shapes that have this radial symmetry about an axis a "volume of revolution." In this description though, you've got to intuit the shape of the cross-section of the resulting solid. Also, since you're basically cutting this solid into a bunch of small disks, finding the volume this way is sometimes called the "disk method," but I think that's dumb. There's also a "shell method," and "washer method," we'll talk about soon.



How about you try one now?

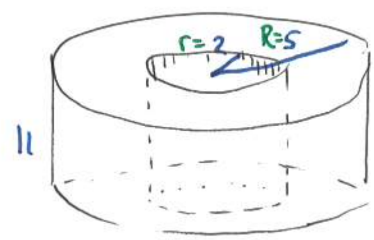
2. Find the volume of the solid formed by taking the area between the curve $y = 1 - x^4$ and the x-axis and rotating it about the x-axis.

1. Find the volume of the solid formed by taking the area under the curve $y = -\frac{1}{2}x + 3$ between $x = 1$ and $x = 4$ and rotating it about the x-axis.

3. (CHALLENGE) Take the same region from #2 and generate a solid by rotating it about the y-axis. What is the volume of that solid?

IntBioCalc Week 2 Thursday

Let's continue with finding the volumes of solids.
Last time the cross-sections were all disks. This time the cross-sections will be different.



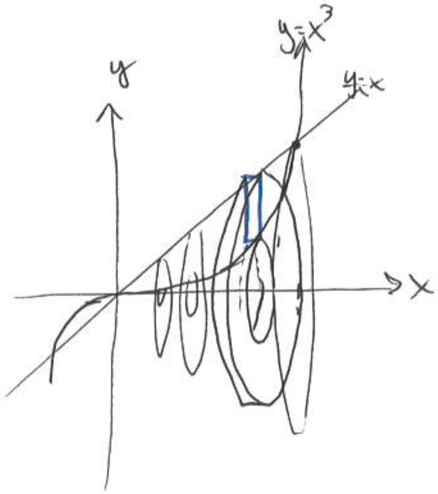
What is the value of this "washer" shape, when the height is 11, the inner radius is r=2, and the outer radius is R=5?

You find the volume by taking the volume of the "whole" cylinder $11 \times \pi(5)^2$ and subtracting our the volume of the "hole", $11 \times \pi(2)^2$. So it's

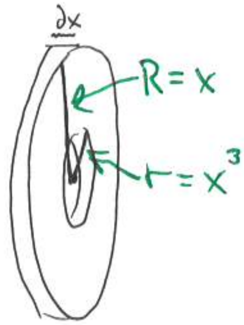
$$11\pi(5)^2 - 11\pi(2)^2 = 11\pi(25-4) = 231\pi$$

Nice! Now we can find the volume of ~~regions~~ solids that have "washer"-shape cross-sections.

What is the volume of the solid formed by taking the region bounded by $y=x$ and $y=x^3$ in the first quadrant and rotating it about the x-axis?



$$\int_0^1 (\text{Area of washer})\, dx$$



$$= \int_0^1 \pi R^2 - \pi r^2 \, dx$$

$$= \pi \int_0^1 (x)^2 - (x^3)^2 \, dx$$

$$= \pi \left(\frac{1}{3}x^3 - \frac{1}{7}x^7 \right) \Big|_0^1 = \pi \left(\frac{1}{3} - \frac{1}{7} \right) = \pi \left(\frac{4}{21} \right) = \frac{4\pi}{21}$$

Now you!

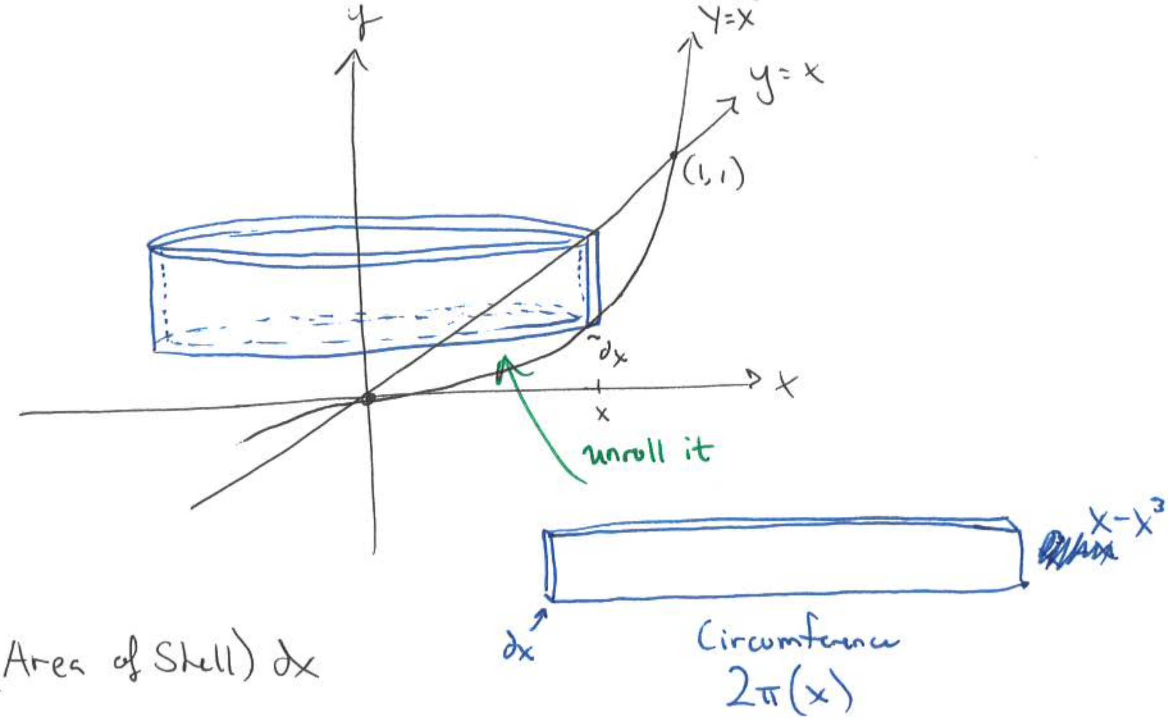
1. Find the volume of the ~~region~~ solid formed by rotating the region bounded by $y=x+1$ $y=\sqrt{x}$ $x=0$ $x=2$ about the x-axis.

$$\pi \int_0^2 (x+1)^2 - (\sqrt{x})^2 \, dx$$

2. Rotate the region bounded by $y=x$ and $y=x^3$ about the y-axis now.

$$\pi \int_0^1 y^{2/3} - y^2 \, dy = \frac{4\pi}{15}$$

Now there's another way to calculate that second volume, rotating the $y=x$ and $y=x^3$ bounded region about the y-axis. This idea is called the shell method. And this method doesn't use cross-sections along the axis, but instead "radial cross-sections" based on a distance from the axis. It's called the "shell method."



$$\text{Volume} = \int_0^1 (\text{Area of Shell})\, dx$$

$$= \int_0^1 (2\pi x)(x - x^3)\, dx$$

$$= 2\pi \int_0^1 x^2 - x^4\, dx$$

$$= 2\pi \left(\frac{1}{3}x^3 - \frac{1}{5}x^5\right)\Bigg|_0^1 = 2\pi\left(\frac{5-3}{15}\right) = \frac{4\pi}{15}$$

Find the volume of the solid formed by rotating the region bounded by $y = \ln(x)$, $y = 2$, and the x-axis and y-axis, about the y-axis.

Now take the same region and rotate it about the x-axis and find the volume.

Again, rotate the region bounded by

$y = \sec\theta$, the x-axis, and $x = -\frac{\pi}{3}$ $x = \frac{\pi}{3}$

about the x-axis and find the volume.