Int Bio Calc Week 3 Monday

Substitution

This week we're talking about methods of evaluating integrals. So far (most) of the integrals I've given you have been straightforward to evaluate: just remember ~~the~~ an antiderivative or think of it as an area and use geometric methods. But finding an antiderivative $\int f\,dx$ for any function f ~~but an~~ might be difficult, or even impossible, depending on f. For example the function e^{-x^2} (the parent function of the NORMAL CURVE used in statistics) doesn't have an antiderivative expressible in terms of elementary functions $\left(x^2, \log(x), \sin(x), \ldots\right)$ In otherwords, you cannot write down an antiderivative

$$\int e^{-x^2}\,dx$$

But even for functions that have a calculatable antiderivative, there is no (easy*) way to do it for any case. So instead we learn techniques of evaluating integrals that may be helpful. The first one we'll learn today, and is called the technique of substitution.

It's also sometimes called change of coordinates, or change of variables, or u-substitution because we often will use u as our new variable. You can think of substitution as "undo-ing the chain rule."

QUESTION – How do you evaluate

$$\int 2x \cos(x^2-1)\, dx \; ?$$

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* The Risch algorithm can compute the antiderivative of an elementary function, but there are caveats, and the description of the algorithm spans 100 pages.

Its tough because there's junk inside the cosine, so Lets substitute for it.

Letting $u = x^2 - 1$ we get $du = 2x\,dx$. We need to use these to translate ALL the xs into u variables

$$\int \cos(x^2-1)\,2x\,dx = \int \cos(u)\,du = \cdots$$

and then we evaluate AND TRANSLATE Back,

$$\cdots = \sin(u) + C = \sin(x^2-1) + C$$

Always translate back to your original variable. And remember that you can always try to substitute for the ugliest thing in your integrand, and it won't ever hurt to ~~thing~~ TRY a substitution (or two or three).

Let's try one with bounds now.

Evaluate $\displaystyle\int_1^9 \frac{1}{\sqrt{x}} e^{-\sqrt{x}}\, dx$.

There are a few options for what to choose as a substitution.

Letting $u = -\sqrt{x}$ we get $du = -\frac{1}{2\sqrt{x}}\, dx$. But in the integrand we only have an extra $\frac{1}{\sqrt{x}}$, so we should write $-2\,du = \frac{1}{\sqrt{x}}\, dx$ to make the change of variables. Now remember too, those original bounds are x bounds. They too need to change to bounds in terms of u.

$$\int_{x=1}^{x=9} e^{-\sqrt{x}} \frac{1}{\sqrt{x}}\, dx = \int_{u=-\sqrt{1}}^{u=-\sqrt{9}} e^{(u)} (-2\,du)$$

$$= -2\int_{-1}^{-3} e^{u}\, du$$

$$= 2\int_{-3}^{-1} e^{u}\, du = 2\left(\frac{1}{e} - \frac{1}{e^3}\right) + C$$

Since definite integrals are just numbers/areas, there's ~~there~~ no ~~need~~ way to change back to the original variable x.

I'll do two more examples then let you practice.

Evaluate $\int \tan(\theta)\, d\theta$.

$= \int \frac{\sin(\theta)}{\cos(\theta)}\, d\theta$. Let $w = \cos\theta$, so $dw = -\sin\theta\, d\theta$.

$$\int \frac{\sin(\theta)}{\cos(\theta)}\, d\theta = -\int \frac{dw}{u} = -\ln(w) + C = -\ln(\cos(\theta)) + C$$
$$= \ln(\sec(\theta)) + C \;//$$

Evaluate $\displaystyle\int_{e^2}^{e^{e^2}} \frac{\ln(\ln(x))}{x\ln(x)}\, dx$

$\mu = \ln(x)$

$\Rightarrow d\mu = \frac{1}{x}\, dx$

So $\displaystyle\int_{e^2}^{e^{e^2}} \frac{\ln(\ln(x))}{\ln(x)} \frac{1}{x}\, dx = \int_{\ln(e^2)=2}^{\ln(e^{e^2})=e^2} \frac{\ln(\mu)}{\mu}\, d\mu = \cdots$

Let $\nu = \ln(\mu)$

so $d\nu = \frac{1}{\mu}\, d\mu$

$$\cdots = \int_{\ln(2)}^{\ln(e^2)=2} \nu\, d\nu = \frac{1}{2}\nu^2 \Big|_{\ln(2)}^{2}$$
$$= \frac{1}{2}\left(4 - \ln(2)^2\right)$$
$$= 2 - \ln(\sqrt{2})^2 \;//$$

$\displaystyle\int_1^2 \frac{x\, dx}{(x^2+1)\ln(x^2+1)}$ $\quad\displaystyle\int_0^2 \frac{x}{x+2}\, dx$

$\displaystyle\int \sqrt{1+\ln(x)}\, \frac{\ln(x)}{x}\, dx$ $\quad\displaystyle\int_2^5 (x-2)\, e^{-\frac{1}{2}(x-2)^2}\, dx$

IntBioCalc   Week 3   Tuesday

Integration by Parts

Today we'll cover another technique of integration called ~~that~~ Integration by parts. The idea behind this one is that it can "undo" the product rule. Remember that for functions u and v of x

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

Integrating both sides of this yields with respect to x

$$\int \frac{d}{dx}(uv)\,dx = \int u\frac{dv}{\cancel{dx}}\,\cancel{dx} + \int v\frac{du}{\cancel{dx}}\,\cancel{dx}$$

$$\Rightarrow \quad uv = \int u\,dv + \int v\,du$$

$$\Rightarrow \quad \int u\,dv = uv - \int v\,du \quad !!!$$

This last equation yields some hope to solve integrals once we realize they have the form $\int u \, dv$ and calculate du and v from it.

EXAMPLE – Evaluate $\int 2x \sin(x) \, dx$.

We want to let u be something that we can simplify by taking a derivative, and dv be something we know the antiderivative of.

Let $u = 2x$ and let $dv = \sin(x) \, dx$.

Then we get $du = 2 \, dx$ and $v = -\cos(x)$.

Then $\int u \, dv = uv - \int v \, du$

$$\int 2x \sin(x) \, dx = -2x\cos(x) - \int (-\cos(x))(2 \, dx)$$

$$\dots = -2x\cos(x) + 2\int \cos(x) \, dx$$

$$= -2x\cos(x) + 2\sin(x) + C$$

$$\cdots = \left(-(3)^2e^{-3} - 2(3)e^{-(3)} - 2e^{-3}\right) - \left(-0 - 0 - 2e^{-0}\right)$$

$$= -9^2e^{-3} - 6e^{-3} - 2e^{-3} + 2 = -17e^{-3} + 2 \;//$$

$$= 2 - \frac{17}{e^3} \;//$$

EXAMPLE – Evaluate $\int \ln(x)\,dx$.

Using "integration by parts" on this one is a tad inspired.

$$u = \ln(x) \qquad dv = dx$$

$$du = \frac{1}{x}\,dx \qquad v = x$$

$$\int \ln(x)\,dx = x\ln(x) - \int x \frac{1}{x}\,dx$$

$$= x\ln(x) - x + C \;//$$

OR

$$= x\left(\ln(x) - 1\right) + C$$

OR

$$= x\ln\left(\frac{x}{e}\right) + C$$

Let's do a kinda weird example.

EXAMPLE - Evaluate $\int e^x \cos(x)\,dx$.

Let $u = e^x$ and $dv = \cos(x)\,dx$

$du = e^x\,dx \qquad v = \sin(x)$

$$\int e^x \cos(x)\,dx = e^x \sin(x) - \int e^x \sin(x)\,dx$$

Doing "by parts" again

$u = e^x \qquad dv = \sin(x)\,dx$

$du = e^x\,dx \qquad v = -\cos(x)$

$$\int e^x \cos(x)\,dx = e^x \sin(x) - \left(-e^x \cos(x) - \int e^x(-\cos(x))\,dx\right)$$

$$\int e^x \cos(x)\,dx = e^x \sin(x) + e^x \cos(x) - \int e^x \cos(x)\,dx$$

But this is right back to where we started!

Clever idea: add $\int e^x \cos(x)\,dx$ to both sides.

$$\int e^x \cos(x)\,dx = e^x\sin(x) + e^x\cos(x) - \int e^x \cos(x)\,dx$$

$$\Rightarrow \quad 2\int e^x \cos(x)\,dx = e^x\sin(x) + e^x\cos(x)$$

$$\Rightarrow \quad \int e^x \cos(x)\,dx = \frac{1}{2}e^x\left(\sin(x) + \cos(x)\right) + C \quad //$$

~~EXAMPLE 2 Evaluate~~ $\cancel{\int \sec(x)\,dx}$.

~~This one is important because its just secant, but notoriously tricky enough to have it's own wikipedia page.~~

$$\int x e^{6x} \qquad \int (3t+5)\cos\left(\frac{t}{4}\right)dt$$

$$\int x^5 \sqrt{x^3+1}\,dx$$

Int Bio Calc Week 3 Wednesday

~~Partial Fraction Decomposition~~

Integrating Rational Functions

Today & tomorrow we'll learn how to integrate rational functions.

Doing this will come down to (re)learning a few algebraic tricks to write the rational function in a nicer form. Recall a rational function looks like

$$\frac{p(x)}{q(x)}$$

for polynomials p and q. For example

$\frac{3x^9-7x^2+1}{x^2-2}$ is a rational function.

The algebraic manipulations we'll need are

- polynomial long division
- partial fraction decomposition
- completing the square.

We'll start with polynomial long division. And we'll do it via taking an integral.

Evaluate $\displaystyle\int \frac{x^4+3x^2+x+7}{x-2}\,dx.$

Notice that (1) factoring the top looks tough, (2) ~~[illegible]~~, ... the degree of the numerator is larger than the degree of the denominator. Maybe (certainly) rewriting this thing after doing long division will help.

$$\begin{array}{r l}
 & x^3 + 2x^2 + 7x + 15 \\
x-2 \,\big) & \overline{x^4 + 0x^3 + 3x^2 + x + 7} \\
 & \underline{-x^4 + 2x^3} \\
 & \quad 2x^3 \\
 & \quad \underline{-2x^3 + 4x^2} \\
 & \qquad\quad 7x^2 + x + 7 \\
 & \qquad\quad \underline{-7x^2 + 14x} \\
 & \qquad\qquad\quad 15x \\
 & \qquad\qquad\quad \underline{-15x + 30} \\
 & \qquad\qquad\qquad 37
\end{array}$$

So $\displaystyle\int \frac{x^4+3x^2+x+7}{x-2}\,dx = \int x^3+2x^2+7x+15+\frac{37}{x-2}\,dx$

$$= \frac{1}{4}x^4 + \frac{2}{3}x^3 + \frac{7}{2}x^2 + 15x + 37\ln(x-2) + C$$

Straightforwad

Now, partial fraction decomposition. Before explaining it in general, we'll do an example.

Consider $\frac{4x+5}{(x-1)(x+2)}$. Taking the integral of that thing might be tough: there's a quadratic in the denominator! You know what would be really ~~nice~~ nice? If there were A, B s.t.

$$\frac{4x+5}{(x-1)(x+2)} \overset{\text{Dream}}{=} \frac{A}{x-1} + \frac{B}{x+2}$$

If such A and B existed, we could combine the fractions on the RHS and get the LHS

$$\frac{4x+5}{(x-1)(x+2)} \overset{\text{Dream}}{=} \frac{A(x+2) + B(x-1)}{(x-1)(x+2)}$$

$$\Rightarrow \quad 4x+5 \overset{\text{Dream}}{=} (A+B)x + (2A-B)$$

Now for our dream to come true, the coefficients on x would have to match up, and so would the constant terms

$$\begin{cases} 4 = A + B \\ \cancel{4}\ 5 = 2A - B \end{cases}$$

This is a system, and we can solve it to find

$$A = 3 \text{ and } B = 1$$

So our ~~the~~ dream has come true!

Evaluate – $\displaystyle\int \frac{4x+5}{(x-1)(x+2)}\,dx$.

$$\int \frac{4x+5}{(x-1)(x+2)}\,dx = \int \frac{3}{x-1}\,dx + \int \frac{1}{x+2}\,dx$$

$$= 3\ln(x-1) + \ln(x+2) + \ln(C)$$

$$= \ln\left(C(x-1)^3(x+2)\right).$$

In general if you have

$$\frac{\text{polynomial of degree less than denominator}}{\text{a bunch of irreducible linear and quadratic factors}}$$

you can write it as a sum of more reasonable rational functions. The goal is to always make the right guess/dream, and then solve for $A, B, C, \ldots$ to make your dream come true.

$$\frac{1}{(x-1)(x^2+x+1)} \overset{\text{Guess}}{=} \frac{A}{x-1} + \frac{Bx+C}{x^2+x+1}$$

$$\frac{1}{(x+1)^3} \overset{\text{guess}}{=} \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{(x+1)^3}$$

$$\frac{1}{(x-1)^2(x+1)(x^2+1)^3} \overset{\text{guess}}{=} \frac{A}{(x-1)} + \frac{B}{(x-1)^2} + \frac{C}{x+1} + \frac{Dx+E}{(x^2+1)} + \cdots$$

$$\cdots + \frac{Fx+G}{(x^2+1)^2} + \frac{Hx+I}{(x^2+1)^3}$$

~~Evaluate these integrals~~

$$\int \frac{1}{(x-1)(x+2)}\,dx = \left(\int \frac{\frac{1}{3}}{x-1}\,dx + \int \frac{-\frac{1}{3}}{x+2}\,dx\right)$$

$$\int \frac{8\cancel{3}x}{(x-1)^2}\,dx = \frac{A}{x-1} + \frac{B}{(x-1)^2}$$

$\cancel{Ax + A + B}$

$Ax - A + B$

$A = \cancel{3}\ 8 \quad B = \cancel{3}\ 8$

$$\int \frac{x^2+x+1}{(x-1)(x+2)(2x+1)}\,dx$$

$$\int \frac{x^3+x^2+x-1}{x-7}\,dx$$

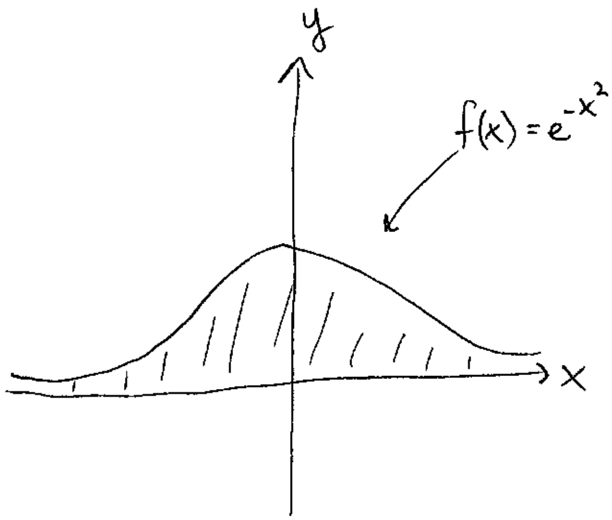
Int Bio Calc Week 3 Thursday

Today I'm going to calculate a single integral for you, and then have you all practice taking integrals yourselves. But first, an aside. Integrals can be tough to evaluate.

Like, if you take some random function f, you might not be able to express it's antiderivative in terms of the elementary functions you know and love, like polynomials or logs or trig.

For example $\int e^{-x^2} dx$ has no closed form.

But that's not to say it's not <u>some</u> function.



Like, that curve nicely bounds area, so $\int_a^b e^{-x^2}\,dx$ exists for any a and b, but in terms of a function F such that $\frac{d}{dx}F = e^{-x^2}$, we can certainly write it as $\int_0^x e^{-t^2}\,dt = F(x) + \underbrace{F(0)}_{C}$, but we can't write it nicely in terms of other functions.

A similar one came up yesterday: $\int \tan(x^2)\,dx$.

~~This~~ A antiderivative of it is certainly a function: $F(x)+C=\int_0^x \tan(t^2)\,dt$, but there's no way to write it down better.

And this is why Riemann sums are important, among other manual/numerical ways to calculate things: sometimes its all we have.

But let's tackle a function that does have a nice antiderivative.

Evaluate $\int \sec\theta \, d\theta$

This integral is infamous, and has its own wikipedia article. It's only right that you see how to calculate it.

Note that

$$\sec\theta = \frac{1}{\cos\theta} = \frac{\cos\theta}{\cos^2\theta} = \frac{\cos\theta}{1-\sin^2\theta}$$

$$= \frac{\cos\theta}{(1-\sin\theta)(1+\sin\theta)}$$

then we can do a substitution, letting $u = 1+\sin\theta$, so $du = \cos\theta \, d\theta$

$$\int \sec\theta \, d\theta = \int \frac{\cos\theta \, d\theta}{(1-\sin\theta)(1+\sin\theta)} = \int \frac{du}{(2-u)(u)}$$

Now we ~~can~~ have a rational function and can break it up via its partial fraction decomposition.

$$\frac{1}{(2-u)u} \overset{\text{dream}}{=} \frac{A}{2-u} + \frac{B}{u}$$

$$\Rightarrow \quad 1 = Au + B(2-u)$$

$$\Rightarrow \quad 0u + 1 = (A-B)u + (2B)$$

$$\Rightarrow \quad B = \frac{1}{2} \quad \& \quad A = \frac{1}{2}$$

So our integral becomes

$$\int \frac{du}{(2-u)u} = \frac{1}{2}\int \frac{du}{2-u} + \frac{1}{2}\int \frac{du}{u}$$

$v = 2-u \quad dv = -du$

$$= -\frac{1}{2}\ln(2-u) + \frac{1}{2}\ln(u) + C$$

$$= \ln\left(\sqrt{\frac{u}{2-u}}\right) + C$$

$$= \ln\left(\sqrt{\frac{1+\sin\theta}{1-\sin\theta}}\right) + C = \ln\left(\sqrt{\frac{(1+\sin\theta)^2}{\underbrace{1-\sin^2\theta}_{\cos^2\theta}}}\right) + C = \ln\left(\frac{1+\sin\theta}{\cos\theta}\right) + C$$

$$= \ln(\sec\theta + \tan\theta) + C$$

$$\int_0^2 \frac{2t}{(t-3)^2}\,\partial t$$

$$\int e^{\sqrt[3]{x}}\,dx \;?$$

tough sub, by parts
love it

$$\int_1^2 \frac{x^2+2}{x+2}$$

$$\int x\sec(x)\tan(x)\,dx$$

$$\int \frac{\ln(\ln(x))}{x}\,dx$$

$$\int \sqrt{\frac{1+x}{1-x}}\,dx$$

$$\int \cos^2\theta\,\partial\theta \qquad \int \cos^3\theta\,\partial\theta$$

$$\int \theta\tan^2\theta\,\partial\theta$$

$$\boxed{\int \sec^3(x)\,dx}$$