Int Bio Calc    Week 5    Monday

## Differential Equations

Mathematically, a <u>differential</u> <u>equation</u> is an equation that relates a function to its derivatives. So something that looks like

$y(t)$ $\quad ay'' + by' + cy = t^2 - 2t$

$y(x)$ $\quad \cos(xy)\, y'' - \sin(y) = 2$

$y(t)$ $\quad \dfrac{\partial y}{\partial t}\, t + e^{y} \dfrac{\partial^2 y}{(\partial t)^2} = y$

$P(\mu, t)$

$$\frac{\partial^2 P}{\partial t\, \partial \mu} = \frac{\partial P}{\partial t} - P\left(\frac{\partial P}{\partial \mu}\right)^2$$

$$F = m\, \frac{\partial^2 P}{\partial t^2}$$

Typically $x$ or $t$ will be your <u>independent</u> variable, and $y$ will be your <u>dependent</u> <u>variable</u> (function).
In this language, you're all been solving DEs in this class.

$$y' = f(t) \iff y = \int f(t)\, \partial t$$

Another basic example is population size $P$ as a function of time $t$. A ~~safe~~ naïve assumption to make is that ~~the~~ "a population will grow at a rate proportional to its size." This indicates we should consider the differential equation $\frac{\partial P}{\partial t} = kP$ for some constant $k$.

This is naïve because eventually a population must "top out," right? Its gotta hit some maximum and $P' = kP$ doesn't take that into account. A more reasonable model for population growth is the logistical model

$$P' = kP\left(1 - \frac{P}{\varepsilon}\right)$$

where $k$ is some constant depending on the population and $\varepsilon$ is this "max out" point called the carrying capacity ($\varepsilon$ for equilibrium). We'll talk more about the real world later. ~~First~~ ~~lets~~ Let's do some math though.

Given a differential equation, ~~involving y as a function of~~ a Solution to that differential equation is any continuous function $y$ that satisfies the equation.

"Verify that $y = \ln(x)$ is a solution to $xy'' + y' = 0$."

But a differential equation might have many solutions.

"Verify that $y = 5\ln(x) + 2$ is a solution to $xy'' + y' = 0$."

A general solution to a differential equation is the family of all solutions to that DE, so it'll involve some constants $C_1$ $C_2$ .... For example $y = C_1 \ln(x) + C_2$ is the general solution to $xy'' + y' = 0$.

Now a particular solution to a differential equation is a single specific function that satisfys the DE and also some initial conditions.

"What is the particular solution to $xy'' + y' = 0$ if we know that $y(1) = 3$ and $y(e) = 5$?"

This is called an initial value problem.

But like I said, you're basically been solving ODEs already by solving integrals.

"What is the general solution to the differential equation $y'(t) = t^2 + 2\cos(t)$?"

And you've also already been solving initial value problems:

"What is the ~~general~~ particular solution to $y'(t) = t^2 + 2\cos(t)$ if we know $y(\frac{\pi}{2}) = \pi^3/24$?"

And you've also already solved basic mathematical modelling problems.

"If the velocity of a bicyclist can be modelled as a function of time by the equation

$$v(t) = t^2 + 2\cos(t)$$

in miles per hour when the cyclist is heading towards/away from you, if the cyclist was $\pi^3/24$ miles from you at $\pi/2$ hours since you started watching them, how far will the cyclist be from you in $2\pi$ hours?"

## Solving Separable ODEs

Solving a differential equation involves taking some integrals at the very least, so like evaluating integrals, solving differential equation can be very hard. But there is one ~~all~~ type of ordinary DE we'll look at that isn't too bad to solve so long as the resulting integrals aren't too bad to evaluate.

A differential equation of first-order is separable if you can write it as

$$N(y)\, y' = M(x)$$

for functions $N(y)$ of $y$ and $M(x)$ of $x$. I.e, if you can "separate" the $x$s from the $y$s.

These DEs ~~eas~~ can be easy to solve since

$$N(y)\frac{dy}{dx} = M(x) \Rightarrow \int N(y)\,dy = \int M(x)\,dx,$$

and you take those integrals. Note that the LHS has a bunch of ys in it, so you may not be able to find an <u>explicit</u> <u>solution</u> $y = \ldots$ to it, but should be able to find at least an <u>implicit</u> solution $n(y) = m(x) + C$. Be sure to make your solution explicit if you can though.

"Solve the following differential equation ~~the~~ $y' = 6y^2x$."

$$y' = 6y^2x$$

$$\Rightarrow \frac{1}{y^2}\,dy = 6x\,dx$$

$$\Rightarrow \int \frac{1}{y^2}\,dy = 6\int x\,dx$$

$$\Rightarrow -\frac{1}{y} + \cancel{C} = 3x^2 + C$$

$$\Rightarrow y = \frac{1}{-3x^2 + C} \;//$$

Note we divided by $y$ here, so we have to consider $y = 0$ separately.

General Solution

" Solve the initial value problem : solve
$y' = 6y^2x$ where $y(1) = 1/9$ . "

Particular solution

$$\frac{1}{9} = \frac{1}{-3(1)^2 + C}$$

$$\Rightarrow \quad 9 = -3 + C \quad \text{so} \quad C = 12$$

$$\Rightarrow \quad y = \frac{1}{12 - 3x^2}$$

We won't worry about domain restrictions :)

Let's look at another.

" Solve $y' = \frac{e^x}{y}$ where $y(0) = 1$. "

$$\int y \, dy = \int e^x \, dx$$

$$\Rightarrow \quad \frac{1}{2}y^2 = e^x + C$$

$$\Rightarrow \quad y = \pm\sqrt{2e^x + C}$$

Since $y(0) = 1$, we'll choose the positive root.

Note $(1) = \pm\sqrt{2e^0 + C} \Rightarrow C = -1$.

$$y = \sqrt{2e^x - 1}.$$

"Solve $y' = xe^{x^2 - \ln(y^2)}$ explicity."

$$y = \sqrt[3]{\frac{3}{2} e^{x^2} + c}$$

"Solve $x^2 y' = \sec(y) - y'$ explicitly."

$$y = \arcsin\left(\arctan(x) + c\right).$$

"Find a particular solution to " if $y(1) = 0$."

"Find the general solution to $\sin^2(y)\, y' = (1 - y')\cos^2(y)$."

"Find the general solution to

$$xy' = \sqrt{x^2 - (xy)^2}$$
."

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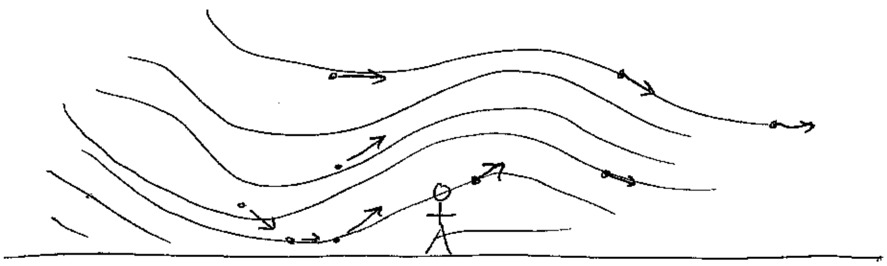
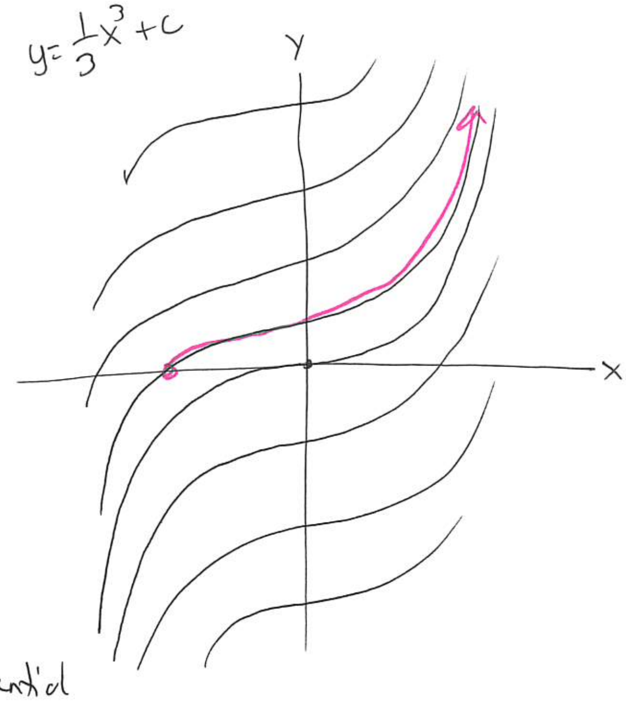
Vector Fields (direction fields)

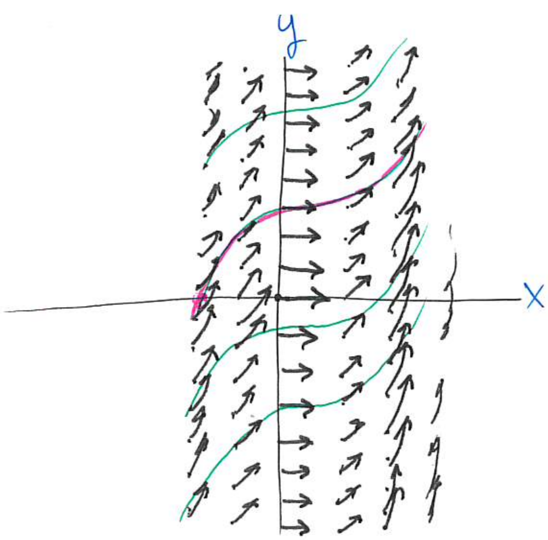
In general a vector field is an associating to each point in some space a vector. The motivating example ~~th~~ to think about is the flow of water in the ocean (currents). Or the direction of wind at any point.



We can use this ~~ea~~ idea to study solutions to first-order differential equations. Suppose we have the differential equation $y' = x^2$. It's general solution will be $y = \frac{1}{3}x^3 + C$. But the idea is that we can see the set of general solutions by looking at a vector field associated to $y' = x^2$.



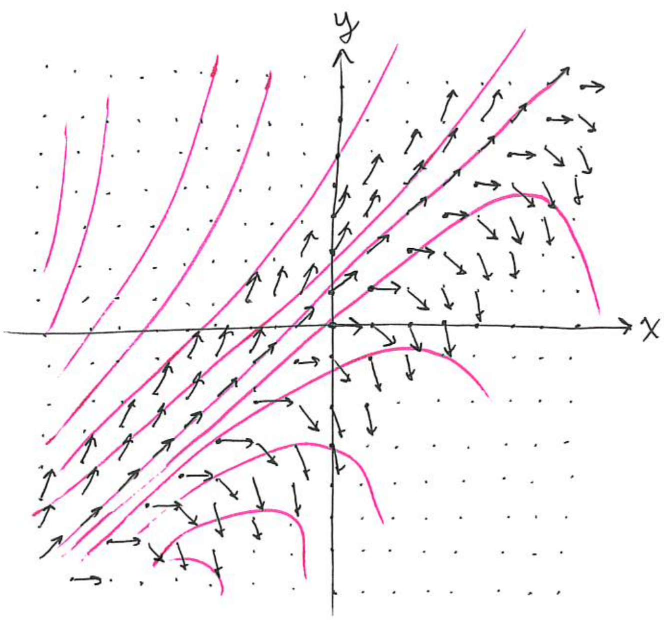

If you remember that $y'$ represents the "rate of change" of $y$ at a point $x$, you might remember that it geometrically corresponds to the slope of the tangent line to the solution $y$ at a point. So if you have a differential equation with a single $y'$ in it, you can solve for $y'$ and find the slope at any point $(x,y)$. Using $y'=x^2$ as an example:




Taking each $(x,y)$ and plugging it in and drawing the slope vector, you get something like this. And you can see the streamlines trace out the solution curves $y$. Imagine ~~you~~ these arrows are tracing out water currents, and you have some initial condition $y(x_0)=y_0$; If you drop a leaf that that intial condition point $(x_0,y_0)$ it'll trace out the particular solution that corresponds to that initial condition. ~~As a result~~ ~~this~~

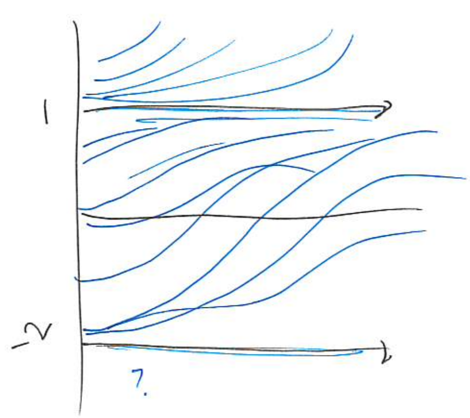
In the example $y' = x^2$, the vector (slope) $y'$ doesn't depend on $y$. Let's sketch a vector field with a little more meat.

"Sketch a vector field for the equation $y' = y - x$,* and sketch the graphs of a few particular solutions."




"Sketch the vector field associated to $y' = (y+2)(1-y)^2$ ~~for~~ on the $(y,t)$-plane for positive $t$."

"Stable solutions"




tougher
$y' = y + x^2$
the curve along which $y'$ is constant is no longer a line :(