Week Five Quiz

Linear Algebra

UCR Math-131, Fall 2020

Recall that the trace of a square matrix A, usually denoted $\text{tr}(A)$, is the sum of the diagonal entries of A.

1. Let $\mathbf{C}$ be the field of complex numbers. The set $\text{Mat}_2(\mathbf{C})$ of square 2×2 matrices with entries in $\mathbf{C}$ is a vector space over $\mathbf{C}$. What is the dimension of this vector space?

   We can explicitly describe $\text{Mat}_2(\mathbf{C})$ in set-builder notation as:

   $$\text{Mat}_2(\mathbf{C}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \; a, b, c, d \in \mathbf{C} \right\}$$

   Since any element of $\text{Mat}_2(\mathbf{C})$ depends freely on a choice of four element of $\mathbf{C}$, this is a four dimensional vector space over $\mathbf{C}$. The most natural choice of basis would be:

   $$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

2. Consider the subset $\mathfrak{sl}_2$ of $\mathrm{Mat}_2(\mathbf{C})$ consisting of matrices with trace equal to zero. Prove that $\mathfrak{sl}_2$ forms a vector subspace of $\mathrm{Mat}_2(\mathbf{C})$. What is the dimension of this subspace $\mathfrak{sl}_2$ over $\mathbf{C}$?

First, $\mathfrak{sl}_2$ forms a subspace of $\mathrm{Mat}_2(\mathbf{C})$ because (1) the zero matrix has trace zero, (2) the sum of two trace zero matrices has trace zero since $\mathrm{tr}(A+B) = \mathrm{tr}(A) + \mathrm{tr}(B)$, and (3) any scalar multiple of a trace zero matrix will have trace zero since $\alpha \mathrm{tr}(A) = \mathrm{tr}(\alpha A)$. Next, since each element of

$$\begin{aligned} \mathfrak{sl}_2 &= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \; a, b, c, d \in \mathbf{C} \text{ and } a + d = 0 \right\} \\ &= \left\{ \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \middle| \; a, b, c \in \mathbf{C} \right\} \end{aligned}$$

can be uniquely determined by a choice of three elements of $\mathbf{C}$, $\mathfrak{sl}_2$ will have dimension three over $\mathbf{C}$. A natural choice of basis for $\mathfrak{sl}_2$ as a vector space over $\mathbf{C}$ is:

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Note for the curious: the notation $\mathfrak{sl}_2$ stands for *special linear*, and typically $\mathfrak{sl}_2$ is thought of as algebraic object called a *Lie algebra*.