Week Seven Quiz

Linear Algebra

UCR Math-131, Fall 2020

1. Write down the linear transformation $\mathbf{R}^2 \to \mathbf{R}^2$ that sends the vector $\left(-\sqrt{2}/2, \sqrt{2}/2\right)$ to $(1,0)$ but fixes the vector $(0,1)$.

   We're looking for a linear transformation of the form $(x,y) \mapsto (ax+by, cx+dy)$ such that

   $$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} -\sqrt{2}/2 \\ \sqrt{2}/2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

   The second of these conditions tells us that $b = 0$ and $d = 1$. Then from the first of these conditions we get

   $$-\frac{\sqrt{2}}{2}a + \frac{\sqrt{2}}{2}b = 1 \quad -\frac{\sqrt{2}}{2}c + \frac{\sqrt{2}}{2}d = 0$$

   and so $a = -\sqrt{2}$ and $c = 1$. Therefore our linear transformation then is

   $$\begin{pmatrix} -\sqrt{2} & 0 \\ 1 & 1 \end{pmatrix}$$

   or $(x,y) \mapsto \left(-\sqrt{2}x, x+y\right)$ if you prefer.

2. Consider the linear transformation $\mathbf{R}^5 \to \mathbf{R}^3$ given by

   $$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 4 & 4 & 4 & 5 \\ -2 & 0 & 2 & 4 & 5 \end{pmatrix}$$

What is the rank of this matrix? What are the dimensions of the range and the null space of this transformation?

> Enumerate the rows of that matrix as R_1 and R_2 and R_3. Notice that $R_3 = 2R_1 - R_2$, but R_1 and R_2 are independent. So the matrix has rank two, which is the same as the dimension of the range, and so the null space must have dimension three by the rank-nullity theorem.