Week Two Quiz

Linear Algebra

UCR Math-131, Fall 2020

1. What is the definition of a vector space over a field?

   A vector space V over a field F is a set of vectors such that $\mathbf{0} \in V$, $\boldsymbol{v} + \boldsymbol{w} \in V$ if $\boldsymbol{v}, \boldsymbol{w} \in V$, and $a\boldsymbol{v} \in V$ if $a \in F$ and $\boldsymbol{v} \in V$.

2. For a vector space V over a field F, prove that if $\boldsymbol{v} \neq \boldsymbol{w} \in V$ and $a \neq 0 \in F$ then $a\boldsymbol{v} \neq a\boldsymbol{w}$.

   Proof For the sake of finding a contradiction, suppose otherwise. Suppose that $\boldsymbol{v} \neq \boldsymbol{w}$ but $a\boldsymbol{v} = a\boldsymbol{w}$. Since F is a field, $a^{-1} \in F$, and this would imply that $a^{-1}a\boldsymbol{v} = a^{-1}a\boldsymbol{w}$, and so $1\boldsymbol{v} = 1\boldsymbol{w}$, and so $\boldsymbol{v} = \boldsymbol{w}$, a contradiction. □

   I worded this as a *proof by contradiction* but it could easily be re-worded as a positive proof of the contrapositive statement: that $a\boldsymbol{v} = a\boldsymbol{w}$ implies either $\boldsymbol{v} = \boldsymbol{w}$ or $a = 0$.

   Proof Suppose that $a\boldsymbol{v} = a\boldsymbol{w}$. If $a = 0$, the statement is true. Otherwise if $a \neq 0$, since F is a field, $a^{-1} \in F$. This implies that $a^{-1}a\boldsymbol{v} = a^{-1}a\boldsymbol{w}$, so $1\boldsymbol{v} = 1\boldsymbol{w}$, and so $\boldsymbol{v} = \boldsymbol{w}$. □