Sample Final Exam

Ordinary Differential Equations

UCR Math-046-E01, Summer 2018

NAME: ____

STUDENT ID: ____________________________________

Please silence your phone during the exam.

Have your ID ready when you turn in your exam to me.

You may use the blank side of any page as scratch paper, but note that I will only be looking for you response to a question on the one side of the page that contains that question.

This exam is long, but that's okay. I don't expect you to respond to *all* the questions in the allotted time. The exam is long so that you have options as to which questions you answer and which you don't answer.

Each page of this exam will be weighted equally when being graded.

Remember that the purpose of this exam is to provide me a document to read so that I may assess how much you've learned in the course, and assess how prepared you are the classes for which this class is a prerequisite. I'm not a grading robot, looking only for right answers. I'm just trying to figure out what you know. So please communicate to me with your responses. Even if you aren't entirely sure about a question, let me know what you *do know* about it. Anything to give me evidence that you've learned something.

1. What is the definition of a *differential equation*?

   A differential equation is an equation that relates a function and its derivatives. It draws a relationship between the values of the function and the various orders at which the function is changing (its rate of change, the rate at which its rate of change is changing, etc).

2. Imagine that your friend is taking this class next quarter and is very confused about the difference between a *general solution* and a *particular solution* to a differential equation. Help your friend. Explain the difference between a *general solution* and a *particular solution* to a differential equation.

   So a solution to a differential equation is a function y that satisfies the differential equation, right? I.e. if you calculate the functions y and y' and y'' and so on, and plug those into the differential equation, it makes that equation true. A differential equation might have *many* solutions though! This is like back in integral calculus class, where we had to add a $+C$ to our antiderivatives because there were many different antiderivatives, one for each value of C.

   The *general solution* to a differential equation will also have a bunch of constants $C_1, C_2, \ldots$ in it, and it is the "most general solution" to the differential equation in the sense that *any* solution can be written as the general solution with some actual numbers chosen for the C_is. A *particular solution* is just one of these choices of C_is, where you get an actual solution instead of a thing with a bunch of unknown constants floating around in it.

   So like, if you're ever given *initial conditions* when you're solving a differential equation, conditions like $y(0) = 7$ and $y'(1) = -\pi$ or whatever, you're being asked for a particular solution. You have to solve for the general solution first with those constants in it (well, until you learn about Laplace transforms), but then these conditions "pin down" a particular solution. You use those initial conditions to solve for the C_is.

3. Give me an example of a single variable function such that the derivative of the function is equal to one more than the square of the function.

   Let y be our single variable function. The hard part is to translate the description of y above into the differential equation

$$\dot{y} = y^2 + 1\,.$$

   This equation is separable.

$$\frac{1}{y^2+1}\dot{y} = 1 \quad \Longrightarrow \quad \int \frac{1}{y^2+1}\,\mathrm{d}y = \int \mathrm{d}x$$
$$\arctan(y) = x + C \quad \Longrightarrow \quad y = \tan(x+C)\,.$$

   We just need *any* example of a solution, so we can pick the one where $C = 0$, and so $y = \tan(x)$ is the function in question. This makes sense since the derivative of $\tan(x)$ is $\sec^2(x)$, and the described differential equation is just the Pythagorean identity

$$\sec^2(x) = \big(\tan(x)\big)^2 + 1\,.$$

4. Give me an example of a single-variable function that is equal to the negative of three times its second derivative.

   Let y be our single variable function. The hard part is to translate the description of y above into the differential equation

$$y = -3\ddot{y} \quad \Longrightarrow \quad \ddot{y} + \frac{1}{3}y = 0\,.$$

   We can solve this differential equation by solving the corresponding characteristic polynomial $\lambda^2 + \frac{1}{3} = 0$. This polynomial has imaginary roots $\pm\mathrm{i}\frac{1}{\sqrt{3}}$, which corresponds to a general solution

$$y = c_1 \sin\left(\frac{t}{\sqrt{3}}\right) + c_2 \cos\left(\frac{t}{\sqrt{3}}\right).$$

   Taking derivative of this function twice, we get $\ddot{y} = -\frac{1}{3}c_1 \sin\left(\frac{t}{\sqrt{3}}\right) + -\frac{1}{3}c_2 \cos\left(\frac{t}{\sqrt{3}}\right)$, which indicates that $c_1 = c_2 = 1$. How nice.

5. Solve one of the following differential equations by whatever method you find that works. Please circle the one you choose to solve.

$$y\dot{y} + y^2 = 3ty\dot{y} + 1 \qquad\qquad y'\cot^2(x) + \tan(y) = 0$$

The differential equation on the left is separable.

$$y\dot{y}+y^2 = 3ty\dot{y}+1 \quad\Longrightarrow\quad (1-3t)y\dot{y} = 1-y^2 \quad\Longrightarrow\quad \frac{y}{y^2-1}\dot{y} = \frac{1}{3t-1}.$$

Integrating on the left can be done with a quick substitution for y^2-1:

$$\frac{1}{2}\ln\left(y^2-1\right) = \frac{1}{3}\ln\left(C(3t-1)\right) \quad\Longrightarrow\quad y^2 = C(3t-1)^{2/3}+1.$$

The differential equation on the right is also separable.

$$y'\cot^2(x) + \tan(y) = 0 \quad\Longrightarrow\quad -\frac{1}{\tan(y)}y' = \frac{1}{\cot^2(x)}$$

$$-\frac{\cos(y)}{\sin(y)}y' = \tan^2(x)$$

$$-\int\frac{\cos(y)}{\sin(y)}\,\mathrm{d}y = \int\sec^2(x)-1\,\mathrm{d}x$$

The integral on the left requires a substitution for sine, but the integral on the right can be done straight-away:

$$-\ln(\sin(y)) = \tan(x) - x + C \quad\Longrightarrow\quad \frac{1}{\sin(y)} = Ce^{\tan(x)-x}$$

$$\csc(y) = Ce^{\tan(x)-x}$$

$$y = \operatorname{arccsc}\left(Ce^{\tan(x)-x}\right).$$

6. Solve one of the following initial value problems by whatever method you find that works. Please circle the one you choose to solve.

$$t\dot{y} - 2y = t^4 \sin(2t) \quad \text{where } y(\pi) = \tfrac{3}{2}(1-\pi)$$

$$y' = 3\tfrac{y}{x} - \tfrac{x}{y} \quad \text{where } y\left(\sqrt{2}\right) = 5 \text{ and } x > 0.$$

The differential equation on the left is first-order linear. After dividing through by t, we multiply through by the integrating factor

$$e^{-\int \frac{2}{t}\,dt} = \frac{1}{t^2}.$$

So our differential equation becomes

$$\frac{1}{t^2}\dot{y} - \frac{2}{t^3}y = t\sin 2t \frac{d}{dt}\left(\frac{1}{t^2}y\right) = t\sin 2t.$$

Integrating both sides (integration by parts on the right), and solving for y we get our solution

$$y = \frac{1}{2}\cos(2t)\left(t^2 - t^3\right) + Ct^2,$$

and our initial condition gives us that $C = 1 - \pi$. We divided through by t initially, so $t = 0$ cannot be part of our domain, and since our initial condition gives us a negative y, the domain of this solution is $(-\infty, 0)$.

The differential equation on the right is homogeneous. Use the substitution $y = xv$ and $y' = xv' + v$.

$$y' = 3\frac{y}{x} - \frac{x}{y} \implies xv' + v = 3v - \frac{1}{v}$$

$$\int \frac{v}{2v^2 - 1}\,dv = \int \frac{1}{x}\,dx$$

$$\ln(2v^2 - 1) = 4\ln(Cx) \implies y^2 = Cx^6 + \frac{1}{2}x^2$$

Using our initial condition, we get that $C = 3$, and so our solution is

$$y = \sqrt{3x^6 + \frac{1}{2}x^2}.$$

Since $3x^6 + \frac{1}{2}x^2 > 0$ for all $x > 0$, the domain of our solution is $(0, \infty)$.

7. Solve one of the following differential equations by whatever method you find that works. Please circle the one you choose to solve.

$$y' + y - y^4 = 0 \qquad\qquad 2xy^2 + 1 + (2x^2y + 1)y' = 0$$

The differential equation on the left is Bernoulli. First rewrite it as

$$y' + y = y^4 \quad \Longrightarrow \quad y^{-4}y' + y^{-3} = 1,$$

then do the substitution where $v = y^{-3}$, and so $v' = -3y^{-4}y'$. The resulting equation is then separable:

$$\frac{1}{3}v' + v = 1 \quad \Longrightarrow \quad \frac{1}{v-1}v' = -3$$
$$\ln(v-1) = -3x + C$$
$$v = Ce^{-3x} + 1$$
$$y^{-3} = Ce^{-3x} + 1 \quad \Longrightarrow \quad y^3 = \frac{1}{Ce^{-3x}+1},$$

The differential equation on the right is exact. We can see this more easily if we rewrite it as

$$(2xy^2 + 1)\mathrm{d}x + (2x^2y + 1)\mathrm{d}y = 0$$

Letting $M = 2xy^2 + 1$ and $N = 2x^2y + 1$, we can see that it's exact since

$$M_y = 4xy = N_x.$$

We need to find the function G such that $G_x = M$ and $G_y = N$. We need

$$G = \int M\,\mathrm{d}x = x^2y^2 + x + f_1(y)$$
$$G = \int N\,\mathrm{d}y = x^2y^2 + y + f_2(x).$$

Comparing these two expressions for G, the function $f_1(y)$ must be only y plus some constant, and $f_2(x)$ must contain just x plus some constant. So the solution to this differential equation is $x^2y^2 + x + y = C$.

8. Solve one of the following differential equations by whatever method you find that works. Please circle the one you choose to solve.

$$\sqrt{x} = \frac{1}{x}\dot{y} - (y\ddot{y})^2 \qquad\qquad \frac{y''' + 3y'}{-3y'' - y} = 1$$

After looking over the differential equation on the left for a moment, you should notice that it looks hard! There's a $\ddot{y}^2$ in there, and we didn't discuss any techniques to handle that. It's not even linear. We should turn our focus to the other differential equation.

The differential equation on the right is more promising. After rearranging it a bit, we get it to look like

$$y''' + 3y'' + 3y' + y = 0$$

which we may solve by looking at the roots of the corresponding characteristic polynomial

$$\lambda^3 + 3\lambda^2 + 3\lambda + 1 = 0\,.$$

This cubic may look familiar to you, so you might already know its factorization. But in case you don't, let's start guessing roots. Always start guessing your favorite numbers: 0, 1, and -1. Notice that -1 is a root, so $(\lambda + 1)$ is a factor We can divide this factor out, which luckily leaves us with a quadratic that we can factor easily:

$$\begin{aligned} \lambda^3 + 3\lambda^2 + 3\lambda + 1 &= (\lambda + 1)\left(\lambda^2 + 2\lambda + 1\right) \\ &= (\lambda + 1)(\lambda + 1)(\lambda + 1) \;=\; (\lambda + 1)^3 = 0\,. \end{aligned}$$

The characteristic polynomial has a repeated root of -1, which indicates that the general solution to this differential equation is

$$y = c_1 \mathrm{e}^{-t} + c_2 t\mathrm{e}^{-t} + c_3 t^2 \mathrm{e}^{-t}\,.$$

9. Find the general solution to the following differential equation by whatever method you find that works.

$$\ddot{y} + y = 3e^x + 2\dot{y}$$

After moving the $2\dot{y}$ over to the left, this is a non-homogeneous linear differential equation. We can solve this. First we've gotta find the solution to the corresponding homogeneous differential equation $\ddot{y} - 2\dot{y} + y = 0$. Its characteristic polynomial has a root of 1 of multiplicity two, so the complementary solution is $y_c = c_1 e^x + c_2 x e^x$.

Since our complementary solution y_c overlaps with the function $3e^x$ on the right-hand-side of the differential equation, we should guess that our particular solution looks something like $Y_p = Ax^2 e^x$. Taking a couple of derivatives of Y_p we get

$$Y_p' = A\left(2xe^x + x^2 e^x\right) \qquad Y_p'' = A\left(2e^x + 4xe^x + x^2 e^x\right).$$

Then plugging these into our differential equation, we get

$$A\left(2e^x + 4xe^x + x^2e^x\right) - 2A\left(2xe^x + x^2e^x\right) + Ax^2e^x = 3e^x$$
$$(2A)e^x \; + \; (4A - 4A)xe^x \; + \; (A - 2A + A)x^2e^x = 3e^x,$$

and immediately see that A must be $\frac{3}{2}$. So our general solution is

$$y = y_c + Y_p \;=\; c_1 e^x + c_2 x e^x + \frac{3}{2} x^2 e^x.$$

10. A tank initially holds 100 gallons of Kool-Aid solution containing 1 lb of dissolved Kool-Aid mix. At $t = 0$ another Kool-Aid solution containing 2 lb of mix *per gallon* is poured into the tank at a rate of 3 gal/min, while the well-stirred mixture leaves the tank at a rate of 8 gal/min. Write down a function $Q(t)$ that returns the amount of Kool-Aid mix in the tank at any time t. At what time does the mixture in the tank contain exactly 2 lb of Kool-Aid mix?

To set up an initial value problem that models this situation, recall that the rate at which Q is changing is the rate at which water is flowing into (respectively out of) the tank times the concentration of Kool-Aid mix in that water. So our differential equation is

$$\dot{Q} = (3)(2) - (8)\left(\frac{Q}{100-5t}\right)$$

$$\implies \quad \dot{Q} + \left(\frac{8}{100-5t}\right)Q = 6 \quad \text{where} \quad Q(0) = 1\,\text{lbs of Kool-Aid mix}.$$

This is a first-order linear differential equation, so we can solve it by multiplying through by the integrating factor $e^{\int p(t)\,dt}$ where $p(t) = \frac{8}{100-5t}$. This integrating factor turns out to be $(t-20)^{-8/5}$. Multiplying through by this integrating factor yields

$$(t-20)^{-8/5}\dot{Q} + (t-20)^{-8/5}\left(\frac{8}{100-5t}\right)Q = 6(t-20)^{-8/5}$$

$$\int \frac{d}{dt}\left(Q(t-20)^{-8/5}\right)dt = \int 6(t-20)^{-8/5}\,dt$$

$$Q(t) = (t-20)^{8/5}\left(6\frac{-5}{3}(t-20)^{-3/5} + C\right)$$

$$Q(t) = \left((200-10t) + C(t-20)^{8/5}\right).$$

What a gross problem this has become. Applying our initial condition that $Q(0) = 1$, we get that $C = -199(-20)^{-8/5}$. So then the time t at which there are exactly 2 lbs of Kool-Aid mix in the tank is given by

$$2 = Q(t) \;=\; (200-10t) + -199(-20)^{-8/5}(t-20)^{8/5},$$

which would just be horrendous to solve, so let's leave it at that.

11. We're studying a group of 300 otters in a creek. The size of the group of otters will grow at a rate that is proportional to its current population. In the absence of any outside factors the population will increase by a factor of 8 every three weeks. On any given day about 20 otters wander off to join another group. Will the group survive and thrive, or eventually all wander off? NOTE: ln(8) is slightly greater than 2.

    The differential equation that we want to use to model this situation is

    $$\dot{P} = kP - N$$

    where k is some constant and N is the number of otters that leave in a given day, which is 20. We need to find k, so for a function $\mathtt{P}$ that ignores the otters that wander off, we know that $8\mathtt{P}(0) = \mathtt{P}(21)$. To use this fact we need to solve the differential equation $\dot{\mathtt{P}} = k\mathtt{P}$. This equation is separable, and the solution is $\mathtt{P} = Ce^{kt}$, so we have

    $$8\mathtt{P}(0) = \mathtt{P}(21) \quad \Longrightarrow \quad 8Ce^{k(0)} = Ce^{k(21)}$$
    $$\frac{1}{21}\ln(8) = k$$

    So our differential equation that models the entire situation becomes

    $$\dot{P} = \frac{\ln(8)}{21}P - 20.$$

    To answer the question, we need to solve this differential equation and see if $P(t)$ is ever zero. This differential equation is separable, so we can solve it:

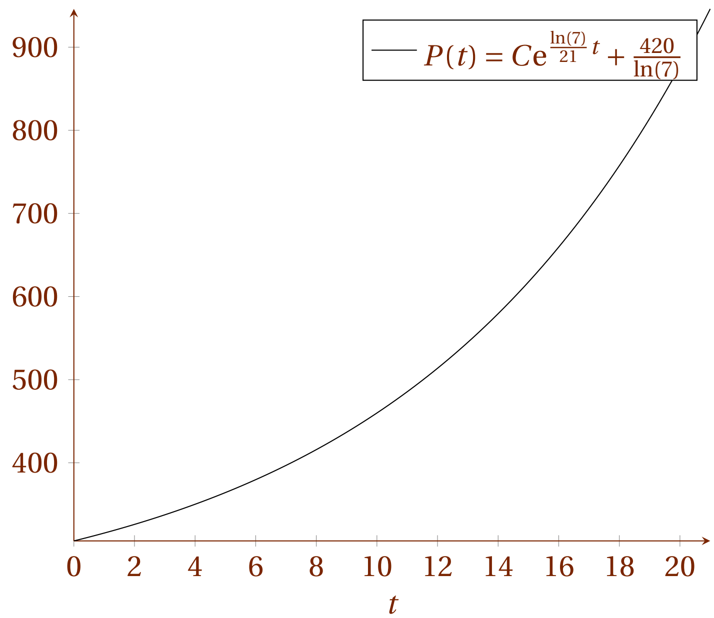
    $$\dot{P} = \frac{\ln(8)}{21}P - 20$$
    $$\int \left(\frac{1}{P - \frac{21}{\ln(8)}20}\right) \mathrm{d}P = \int \frac{\ln(8)}{21}\,\mathrm{d}t$$
    $$\ln\left(P - \frac{420}{\ln(8)}\right) = \frac{\ln(8)}{21}t + C$$
    $$P(t) = Ce^{\frac{\ln(8)}{21}t} + \frac{420}{\ln(8)}$$

Then we must use our initial condition that $P(0) = 300$ to solve for C

$$300 = Ce^{\frac{\ln(8)}{21}(0)} + \frac{420}{\ln(8)}$$

$$300 - \frac{420}{\ln(8)} = C$$

in particular since $\ln(8) > 2$ as per the NOTE, C will be a positive number, and the solution function $P(t)$ will grow at t increases:



So according to the initial model the size of the group of otters will continue to grow and the otters will thrive in the group.

12. Suppose that e^{3x} and $\frac{1}{x}$ are linearly independent complementary solutions to the differential equation

$$y'' + \frac{9x^2+2}{3x^2+x}y' - \frac{54x^2+9x+6}{3x^2+x}y = x\ln(x) \qquad x>0\,.$$

Using the method of *variation of parameters*, calculate the general solution to this differential equation. You may write the general solution in terms of one or more indefinite integrals.

Let $y_1 = e^{3x}$ and $y_2 = \frac{1}{x}$. One could simply recall from the textbook or the homework the formula that we have for the particular solution that is in terms of the Wronskian of y_1 and y_2. The one that looks like

$$Y_p = -y_1 \int \frac{g\,y_2}{W(y_1,y_2)}\,dt + y_2 \int \frac{g\,y_1}{W(y_1,y_2)}\,dt\,.$$

But if we don't remember that, we can proceed by guessing that our particular solution will look like $Y_p = \mu_1 y_1 + \mu_2 y_2$ for function μ_1 and μ_2, and solve the system

$$\begin{cases} \dot\mu_1 y_1 + \dot\mu_2 y_2 = 0 \\ \dot\mu_1 \dot y_1 + \dot\mu_2 \dot y_2 = g(x) \end{cases} \qquad = \qquad \begin{cases} \dot\mu_1 e^{3x} + \dot\mu_2 \frac{1}{x} = 0 \\ 3\dot\mu_1 e^{3x} - \dot\mu_2 \frac{1}{x^2} = x\ln(x) \end{cases} \quad .$$

Doing this, I get that

$$\dot\mu_1 = \frac{x^2\ln(x)}{3x+1}e^{-3x} \qquad \dot\mu_2 = -\frac{x^3\ln(x)}{3x+1}\,,$$

and so looking back at our guess, our particular solution must be

$$Y_p = e^{3x}\int \frac{x^2\ln(x)}{3x+1}e^{-3x}\,dx + \frac{1}{x}\int -\frac{x^3\ln(x)}{3x+1}\,dx\,.$$

But the question asks us for the *general* solution! So we must add this to the original complementary solution. The general solution is then

$$y = c_1 e^{3x} + c_2\frac{1}{x} + e^{3x}\int \frac{x^2\ln(x)}{3x+1}e^{-3x}\,dx + \frac{1}{x}\int -\frac{x^3\ln(x)}{3x+1}\,dx\,.$$

13. For a function of $f(t)$, what is the definition of the Laplace transform $\mathscr{L}\{f(t)\}$?

$$\mathscr{L}\{f(t)\} := \int_0^\infty e^{-st} f(t)\,dt$$

The next part isn't too important, but we should remember that there are some conditions that f must satisfy for $\mathscr{L}\{f(t)\}$ to exist, otherwise that integral might not converge. The Laplace transform $\mathscr{L}\{f(t)\}$ exists if f is of *exponential order* α for some $\alpha > 0$, which means that there exists some $M > 0$ and $x_0 > 0$ such that $|f(t)| \leq Me^{\alpha x}$ for all $x > x_0$.

14. Given the initial value problem of solving $y' - 7y = \sinh(3x)$ where $y(0) = 42$, find an expression for $\mathscr{L}\{y\}$. I'll write some (possibly) helpful Laplace transforms on the board.

We've gotta realize that we need the Laplace transforms

$$\mathscr{L}\{y'\} = s\mathscr{L}\{y\} - y(0) \qquad \mathscr{L}\{\sinh(ax)\} = \frac{a}{s^2 - a^2}.$$

Knowing these, we are unstoppable:

$$\begin{aligned}
\mathscr{L}\left\{y' - 7y\right\} &= \mathscr{L}\{\sinh(3x)\} \\
\mathscr{L}\left\{y'\right\} - 7\mathscr{L}\left\{y\right\} &= \mathscr{L}\{\sinh(3x)\} \\
\left(s\mathscr{L}\left\{y\right\} - y(0)\right) - 7\mathscr{L}\left\{y\right\} &= \frac{3}{s^2 - 9} \\
(s-7)\mathscr{L}\left\{y\right\} - 42 &= \frac{3}{s^2 - 9} \\
\mathscr{L}\left\{y\right\} &= \frac{3}{(s-7)(s^2-9)} + \frac{42}{(s-7)}
\end{aligned}$$

15. Compute the Laplace transform of the following function. I will write some (hopefully) helpful Laplace transforms on the board.

$$g(x) = \begin{cases} 14x + \sin(x) - 49 & 0 \le x < 7 \\ \sin(x) + x^2 & 7 < x \end{cases}$$

First we use a Heaviside function to write this as a single expression:

$$\begin{aligned} g(x) &= 14x + \sin(x) - 49 + u_7(x)\big(\sin(x) + x^2 - (14x + \sin(x) - 49)\big) \\ &= 14x + \sin(x) - 49 + u_7(x)\big(x^2 - 14x + 49\big) \\ &= 14x + \sin(x) - 49 + u_7(x)\big((x-7)^2\big) \end{aligned}$$

And then we can take the Laplace transform of the whole thing:

$$\begin{aligned} \mathscr{L}\left\{g(x)\right\} &= \mathscr{L}\left\{14x + \sin(x) - 49 + u_7(x)\big((x-7)^2\big)\right\} \\ &= 14\mathscr{L}\{x\} + \mathscr{L}\{\sin(x)\} - 49\mathscr{L}\{1\} + \mathscr{L}\left\{u_7(x)(x-7)^2\right\} \\ &= 14\frac{1}{x^2} + \frac{1}{s^2+1} - 49\frac{1}{s} + e^{-7s}\mathscr{L}\left\{x^2\right\} \\ &= 14\frac{1}{s^2} + \frac{1}{s^2+1} - 49\frac{1}{s} + e^{-7s}\frac{2}{s^3}. \end{aligned}$$

And since I'm in a mood to try to make it a little prettier,

$$\mathscr{L}\left\{g(x)\right\} = \frac{2e^{-7s} - 49s^2 + 14s}{s^3} + \frac{1}{s^2+1}.$$