Homework 3

Ordinary Differential Equations

UCR Math-046-E01, Summer 2018

1. Make a substitution to turn the differential equation $y' = \mathrm{e}^{5y+2x} - 1$ into an equivalent separable differential equation. Bonus points if you go on from there to write down the general solution. (CRYPTIC HINT: $\frac{1}{\mathrm{e}^t-1} = \frac{\mathrm{e}^{-t}}{1-\mathrm{e}^{-t}}$)

2. Write down the partial derivatives f_x and f_y for the following functions $f(x,y)$, where we're thinking of y as a variable and *not* as a function of x.

   (a) $f(x,y) = 4x + 2y^2$

   (b) $f(x,y) = \sin(y + xy) + \ln(xy)$

   (c) $f(x,y) = 7y^3\mathrm{e}^{2x} - \frac{1}{x^2y^2}$

   This exercise is really for you to practice, (especially if you haven't seen this before) so I've written down the solutions on the back for you to check with.

3. Show that the following differential equation is not exact:

$$(y+1)\,\mathrm{d}x + (-x)\,\mathrm{d}y = 0$$

   Now multiply the equation through by an *integrating factor* of x^{-2}, and show that the resulting differential equation *is* exact. You don't need to solve the resulting differential equation, but I wanted you to see that *sometimes* there is an integrating factor that can be used to make a non-exact equation exact.

4. It should be clear from the way that the differential equation

$$y' + \cos(x^2)\,y = \mathrm{e}^x y \tag{$*$}$$

   is written that it is a *Bernoulli* differential equation (albeit a silly one, since the power on the right-most y is 1).

   (a) Recall the form of a *first-order linear* differential equation, and rewrite the equation $(*)$ in this form.

   (b) Recall the form of a *separable* differential equation, and demonstrate that $(*)$ is also *separable*.

(c) Furthermore, demonstrate that $(*)$ is also an *exact* differential equation. (In fact, every separable differential equation is exact. Do you see why?)

The purpose of this exercise was to show you that a differential equation doesn't necessarily fall into a single category of being separable, or Bernoulli, or exact, or homogeneous, etc, and that there may not be a single correct method to solve a given differential equation. When solving differential equations, you may use whatever method that works, and multiple methods might work.

5. Solve the following differential equations using whatever methods you find that work. If there is an initial condition on y with the differential equation, find the particular solution corresponding to that initial condition. Don't forget to specify the domain on which that particular solution lives. And remember that $\dot{y}$ is just another notation for y'.

   (a) $2xy^2 + 4 = 2(3 - x^2 y)\dot{y}$

   (b) $e^{yy'} = t$

   (c) $y' = 5y + \mathrm{e}^{-2x} y^{-2}$

   (d) $\cos(x)y' + \sin(x)y = 2\cos^3(x)\sin(x) - 1$
   where $y\left(\frac{\pi}{4}\right) = 3\sqrt{2}$ and $0 \le x < \frac{\pi}{2}$.

   (e) $\frac{x^3}{xy^2 - y^3} = y'$

   (f) $\left(2y\mathrm{e}^{3ty} + 3ty^2\mathrm{e}^{3ty}\right) y' = 1 - 3y^3\mathrm{e}^{3ty}$ where $y(0) = 1$.
   Finding an *explicit* solution to this one is impossible, so unfortunately you must be satisfied with an *implicit* solution.

   (g) $4tyy' = y^2 + t^2$

   (h) $\dot{y} = \cos^2(y)(\mathrm{e}^x - \mathrm{e}^{-x})$

Solutions to question 2.

(a) $f_x = 4 \qquad f_y = 4y$

(b) $f_x = y\cos(y + xy) + \frac{1}{x} \qquad f_y = (1 + x)\cos(y + xy) + \frac{1}{y}$

(c) $f_x = 14y^3\mathrm{e}^{2x} + \frac{2}{x^3y^2} \qquad f_y = 21y^2\mathrm{e}^{2x} + \frac{2}{x^2y^3}$