Homework 4

Ordinary Differential Equations

UCR Math-046-E01, Summer 2018

1. Suppose you throw a 3 kg watermelon off the top of a tall building downward towards the parking lot below with an initial velocity of 17 m/s. While falling, the force of air resistance on your watermelon is 3 times the velocity of the falling melon. Write down a function that returns the velocity $v(t)$ of the watermelon after t seconds of being thrown. How long after being thrown will the watermelon be traveling at 11 m/s?

   HINT: Recall that acceleration due to gravity near the surface of the earth is given by $9.81\,\mathrm{m/s^2}$, but I think for the sake of making calculations easier you should just round that number to $10\,\mathrm{m/s^2}$.

2. A stone having a mass of 2 lbs is dropped from a bridge with no initial velocity, and encounters air resistance that is exactly equal to the square of its velocity. What is the velocity of the stone after 1 minute? For this question use imperial units, so acceleration due to gravity is $32\,\mathrm{ft/s^2}$ instead of the usual metric $9.81\,\mathrm{m/s^2}$.

3. A ball of mass 3 grams is thrown vertically into the air with an initial velocity 12 m/s. Suppose the ball encounters an air resistance equal to 5 times its velocity. Find a function $v(t)$ that returns the velocity of ball at a given time t. How long after being thrown upward does the ball reach its maximum height?

   HINT: Recall that acceleration due to gravity near the surface of the earth is given by $9.81\,\mathrm{m/s^2}$, but I think for the sake of making calculations easier you should just round that number to $10\,\mathrm{m/s^2}$. Also, this problem is kinda tough. Since the direction of the ball changes, the velocity changes sign, so you should write your force of air resistance as $5|v|$, and you should use $\ln|x|$ with absolute value bars if it comes up. Throughout the problem, whenever you have to make a choice about whether contents of the absolute value should be positive or negative, make sure your choice makes sense when relating back to the physical situation. Like, time t should always be positive, and since the ball eventually falls, for large values of t the sign of velocity v should match the sign you've assigned to the downward direction. Try your best! Looking at the later examples in Paul's Notes on Modeling may help with this problem.

4. Suppose that there is a plane flying over the earth, equipped with a cannon that is pointed towards the earth. The cannon shoots a 2 kg cannonball at earth at an initial speed of 300 ft/s. But this is no ordinary cannonball. This is a Smart Cannonball™. After being fired it will gradually reconfigure its shape to become more aerodynamic to reduce air resistance. Following the convention that forces acting on the cannonball in the direction away from earth are negative, the Smart Corporation estimates that after t seconds of being fired the force from air resistance exerted on the cannonball will be $-\frac{v}{1+t}$, where v is the velocity of the cannonball at time t.

   Using this new estimate for the force of air resistance on the cannonball, write down a differential equation that models the motion of the cannonball. Then find a function $v(t)$ that returns the velocity of the cannonball after t seconds of being fired.

5. For you solution $v(t)$ in question 4, calculate the limit $\lim_{t\to\infty} v(t)$. Does the differential equation you developed in the previous question have any equilibrium solutions? Based on this information, what can you conclude about the accuracy of Smart Corporation's claim that the force from air resistance will be given by $-\frac{v}{1+t}$ after t seconds?

6. Consider a tank used in certain hydrodynamic experiments. After one experiment the tank contains 200 liters of a dye solution with a concentration of 1 g/liter. To prepare for the next experiment, the tank is to be rinsed with fresh water flowing in at a rate of 2 liters/min, and the well-stirred solution will flow out at the same rate. How much time will have to elapse before the concentration of the dye in the tank reaches 1% of its original value?

7. Consider the following physical situation:

   > A 1000 gallon holding tank that catches runoff from some chemical process initially has 800 gallons of water with 2 ounces of pollution dissolved in it. Polluted water flows into the tank at a rate of 3 gal/hr and contains 5 ounces/gal of pollution in it. A well mixed solution leaves the tank at 3 gal/hr as well. When the amount of pollution in the holding tank reaches 500 ounces the inflow of polluted water is cut off and fresh water will enter the tank at a decreased rate of 2 gal/hr while the outflow is increased to 4 gal/hr.

   This physical situation requires *two* differential equations, each with their own initial condition to properly model it: one for before the time t_m when pollution in the mixing tank reaches is maximum allowed amount at 500 ounces, and another for between t_m and the time when the tank runs dry. Try to write down the differential equations and initial conditions that model this situation. Once you've tried to do this on your own, go read through Example 2 in Paul's Notes on Modeling.

8. A flock of turkeys in a region will grow at a rate that is proportional to its current population. In the absence of any outside factors the population will triple in 5 days. On any given day about 3 turkeys die of natural causes, 9 turkeys are taken by hunters, and 2 turkeys wander into the flock from neighboring regions. If there are initially 50 turkeys in the flock will the flock survive or die out?

9. Suppose that the size of a colony of ants in a nest that is casually being marauded by anteaters can be modeled by the initial value problem

$$\dot{P} = -3\left(\tan(3t) + \frac{P}{3}\right)P \quad \text{where } P(0) = 1$$

where $P(t)$ denotes the population of ants in hundreds at t weeks since the anteater attack began. How long will it take for the anteaters to eat all the ants in the colony?

10. Some wildlife conservationists want to reintroduce flamingos to an uninhabited region where flamingos once thrived before being wiped out from overhunting. After introducing an initial population of flamingos to the region, the conservationists suspect that the differential equation

$$\dot{P} = -\frac{1}{2}\left(P^3 - 4P^2 + 3P\right)$$

will be an accurate model for the population of flamingos over time, where $P(t)$ is measured in thousands of flamingos after t years of introducing the flamingos. What is the *carrying capacity* of the population according to this differential equation? According to the model, how large does the initial population need to be to ensure that the population survives? (HINT: consider the vector field.)

11. Read Paul's Notes on Euler's Method to gain a basic familiarity of what Euler's Method of approximating solutions to differential equations is and how it works. Don't spend too much time on this; just basic familiarity. Since this is such a numerical/algorithmic thing, I think it's easier to learn about by reading about it rather than listening to me lecture on it. For homework, write down a brief summary of it.