Homework 5

Ordinary Differential Equations

UCR Math-046-E01, Summer 2018

1. For each of the following claims, decide if the claim is true or false. It the claim is false, state why it's false.

   (a) $y(x) = \mathrm{e}^{2x}$ is a solution to $y'' - y' - 6y = 0$.

   (b) $y(x) = 2\mathrm{e}^{x} - \mathrm{e}^{2x}$ is a solution to $y''' - 2y'' - y' + 2y = 0$.

   (c) $y(x) = \sin(\mathrm{i}x)$ is a solution, although not a *real* solution, to the differential equation $y''' - 2y'' - y' + 2y = 0$, where $\mathrm{i}^2 = -1$. HINT: Don't be afraid of the number i; when taking derivatives i behaves just like any other constant.

   (d) $y(x) = c_1\mathrm{e}^{x} + c_2\mathrm{e}^{2x}$ is the general solution to $y''' - 2y'' - y' + 2y = 0$.

   (e) Supposing that the two functions $y_1(t) = \mathrm{e}^{-t}$ and $y_2(t) = \sin(t)$ are solutions to some differential equation, they are also linearly independent solutions.

2. (TRICKY) For each of the following sets of three functions, demonstrate directly that they are *not* linearly independent. That is, use the definition of linear independence instead of calculating the Wronskian. The Wronskian will be gross for these anyways.

   (a)
   $$y_1(x) = \ln(x) + 4 \qquad y_2(x) = x^2 - \ln(x) \qquad y_3(x) = (x+2)^2 - 4x$$

   (b)
   $$f_1(x) = -\frac{1}{2}\sin^2(x) + 3 \qquad f_2(x) = 2\cos(2x) + 2 \qquad f_3(x) = 45$$

3. Demonstrate that if $\lambda_1 \neq \lambda_2$ then the two functions

   $$f_1(t) = \mathrm{e}^{\lambda_1 t} \qquad f_2(t) = \mathrm{e}^{\lambda_2 t}$$

   are linearly independent. This is just a much more general version of an example we did in class, but it's important to realize that this is true in general since we will be dealing a lot with solutions of the form $\mathrm{e}^{\lambda t}$.

4. Find the general solution to the following differential equations by whatever method you find that works. Recall that $\ddot{y}$ is just notation for y'', and similarly $\dddot{y}$ means y'''. If a set of initial conditions is listed, find the particular solution corresponding to those initial conditions. I realize this is a long list. I want you to see a lot of examples that cover most of the different subtle things that can happen when solving linear homogeneous differential equations with constant coefficients. Hopefully each of the differential equations with constant coefficients shouldn't take too much time to solve. Remember that you can check your solutions with good ol' WolframAlpha.

   (a) $\ddot{y} - 5\dot{y} + 6y = 0$ where $y(0) = 0$ and $\dot{y}(0) = -1$

   (b) $y''' = y' + 2y - 2y''$

   (c) $0 = y'' - 2y' - 2y$

   (d) $y''' + y'' = 6y'$ where $y(0) = 0$ and $y'(0) = -7$ and $y''(0) = 1$

   (e) $y'' + 2y' + 10y = 0$

   (f) $\ddot{y} = 49y$ where $y(0) = 49$ and $\dot{y}(0) = -7$

   (g) $\dddot{y} + \dot{y} = \ddot{y} + y$

   (h) $y''' + y'' = y' + y$

   (i) $t^2y'' - t(t+2)y' + (t+2)y = 0$ where $t > 0$, and $y_1(t) = t$ is a solution

   (j) (Tough) $(x-1)y'' - xy' + y = 0$ where $x > 1$, and $y_1(x) = e^x$ is a solution

5. (Optional Extra Practice) Each of the following differential equations is of one of the types we discussed in the first few weeks of class: separable, homogeneous, exact, first-order linear, or Bernoulli. Identify which kind of differential equation each is, and if it's separable, separate the variables and write the differential equation as two equal integrals. I'll put my solutions to this question on the last page of this document.

   (a) $t + \frac{t}{y} = y'$

   (b) $\dot{y} = 3 + xy$

   (c) $\dot{y} - \sin^2(x)\dot{y} = yx^5$

   (d) $e^2\dot{y} + \ln 2^{xy} + e^x(xy)^5$

   (e) $\frac{e^y y'}{x+1} = \frac{xy+x}{x^{-2}}$

   (f) $\frac{y'}{y+t} = 42$

   (g) $yy' + y^2 = 4xyy' + 1$

   (h) $y' = \frac{y\cos(xy)+2xy}{x\cos xy + x^2}$

   (i) $y^2 \ln\left(\frac{y}{x}\right) + xy\dot{y} = (x-y)(x+y)$

   (j) $\ln(x^y) = (x^2-1)y'$

6. Take a break and read a little about the life and legacy of Józef Maria Hoëne-Wroński, the namesake of the Wronskian.

7. (CHALLENGE QUESTION) Recall the theorem about the Wronskian that, when stated for only two functions f and g, goes something like:

   > **Theorem**: If the functions f and g *are a set of solutions* to a second-order linear homogeneous differential equation, then f and g are linearly independent *if and only if* their Wronskian is nonzero.

   Notice the emphasized parts of the theorem. This is an "*if and only if*" statement, which is logically a combination of two *if-then* statements

   - If the functions are linearly independent then their Wronskian is nonzero.
   - If their Wronskian is nonzero then the functions are linearly independent.

   Also one of the hypothesis of the statement is that the functions we're starting with are the solutions to a certain differential equation.

   So a natural question to ask: what if we start with just some arbitrary functions that aren't necessarily the solutions to an second-order linear homogeneous differential equation? Does the same theorem hold? It turns out, NO! Only one of the *if-then* parts of the theorem holds:

   > **Theorem**: For arbitrary differentiable functions f and g, if the Wronskian of those function is nonzero, then the functions f and g are linearly independent.

   The converse of this statement (the other *if-then* part) is not necessarily true! Can you tell me why the converse if not necessarily true? I.e. can you find an example of functions f and g that are linearly independent, but such that their Wronskian is zero?

My solutions to Exercise 5:

(a) separable:
$$\int \frac{y}{y+1}\,\mathrm{d}y = \int t\,\mathrm{d}t$$

(b) linear

(c) separable:
$$\int \frac{1}{y}\,\mathrm{d}y = \int \frac{x^5}{\cos^2(x)}\,\mathrm{d}x$$

(d) Bernoulli

(e) separable:
$$\int \frac{e^y}{y+1}\,\mathrm{d}y = \int x^3(x+1)\,\mathrm{d}x$$

(f) first-order linear

(g) separable:
$$\int \frac{y}{1-y^2}\,\mathrm{d}y = \int \frac{1}{1-4x}\,\mathrm{d}x$$

(h) exact

(i) homogeneous, then separable:
$$\int \frac{v}{1-v^2(2+\ln(v))}\,\mathrm{d}v = \int \frac{1}{x}\,\mathrm{d}x$$

(j) separable:
$$\int \frac{\ln(x)}{x^2-1}\,\mathrm{d}x = \int \frac{1}{y}\,\mathrm{d}y$$