Homework 6

Ordinary Differential Equations

UCR Math-046-E01, Summer 2018

1. For the nonhomogeneous linear differential equations below, what should be your guess for the particular solution Y_p? Don't bother solving for the undetermined coefficients or anything. I just want you to get a lot of practice at making the right guess. Each of these equations below is taken from Examples 8 and 10 in Paul's Notes on Undetermined Coefficients, so you are encouraged to check your answers there.

   (a) $y'' + 2y' + 1 = 4\cos(6t) - 9\sin(6t)$

   (b) $y'' + 2y' + 1 = -2\sin(t) + \sin(14t) - 5\cos(14t)$

   (c) $y'' + 2y' + 1 = e^{7t} + 6$

   (d) $y'' + 2y' + 1 = 6t^2 - 7\sin(3t) + 9$

   (e) $y'' + 2y' + 1 = 10e^t - 5te{-}8t + 2e{-}8t$

   (f) $y'' + 2y' + 1 = t^2\cos(t) - 5t\sin(t)$

   (g) $y'' + 2y' + 1 = 5e^{-3t} + e{-}3t\cos(6t) - \sin(6t)$

   (h) $y'' + 3y' - 28y = 7t + e^{-7t} - 1$

   (i) $y'' - 100y = 9t^2e^{10t} + \cos(t) - t\sin(t)$

   (j) $4y''' + y = e^{-2t}\sin\left(\frac{t}{2}\right) + 6t\cos\left(\frac{t}{2}\right)$

   (k) $4y'' + 16y' + 17y = e^{-2t}\sin\left(\frac{t}{2}\right) + 6t\cos\left(\frac{t}{2}\right)$

   (l) $y'' + 8y' + 16y = e^{-4t} + (t^2 + 5)e^{-4t}$

2. Find the general solution to the differential equation

$$\ddot{y} = 9x^2 + 2x - 1$$

   using the method of undetermined coefficients. Note that you should get the same solution as you'd get if you integrated each side of the equation twice with respect to x.

3. Solve this differential equation by method of variation of parameters without using the technique developed in the next exercise

$$y'' - 2y' + y = \frac{e^x}{x^5}$$

4. There's a *slightly* easier way to get the solution to a differential equation of the form $\ddot{y} + q(t)\dot{y} + r(t)y = g(t)$ via the method of variation of parameters. We can find a formula that give us the solution instead of solving a system of equations each time. This exercise will walk you through deriving this formula.

   First, recall that our general method of solving such a differential equation is to guess that our particular solution looks like $Y_p = \mu_1 y_1 + \mu_2 y_2$ where μ_1 and μ_2 are functions that satisfy the system

$$\begin{cases} \dot{\mu}_1 y_1 + \dot{\mu}_2 y_2 = 0 \\ \dot{\mu}_1 \dot{y}_1 + \dot{\mu}_2 \dot{y}_2 = g(t) \end{cases}, \qquad (*)$$

   and where y_1 and y_2 are particular solutions to the corresponding homogeneous differential equation (i.e. $y_c = c_1 y_1 + c_2 y_2$).

   (a) We usually just plug in our specific y_1, y_2, and $g(t)$ before solving for $\dot{\mu}_1$ and $\dot{\mu}_2$, but this isn't necessary. Solve the system $(*)$ for $\dot{\mu}_1$ and $\dot{\mu}_2$ in terms of general y_1, y_2, and $g(t)$.

   (b) In each of your expressions for $\dot{\mu}_1$ and $\dot{\mu}_2$, you hopefully notice that the quantity $y_1\dot{y}_2 - \dot{y}_1 y_2$ appears (or if it doesn't, you should be able to make it appear my rearranging slightly). This quantity is simply the *Wronskian* of solutions y_1 and y_2, which we can denote nicely as $\mathrm{W}(y_1, y_2)$.

   So now we recall that we want μ_1 and μ_2, so by integrating your expressions for $\dot{\mu}_1$ and $\dot{\mu}_2$, and plugging those into our guess for the particular solution $Y_p = y_1\mu_1 + y_2\mu_2$, you should get that

$$Y_p = -y_1 \int \frac{g y_2}{\mathrm{W}(y_1, y_2)}\, \mathrm{d}t + y_2 \int \frac{g y_1}{\mathrm{W}(y_1, y_2)}\, \mathrm{d}t$$

   This gives you a nice general formula for the particular solution to the differential equation $\ddot{y} + q(t)\dot{y} + r(t)y = g(t)$. Then adding this to y_c you get your general solution.

5. (CHALLENGE PROBLEM) We've been talking about using the method of variation of parameters to find solutions to *second order* linear differential equations. As I mentioned in class, we can also apply this method to solve a linear differential equation of *any* order. It just comes down to solving the system of equations

$$\begin{cases} \mu_1' y_1 + \mu_2' y_2 + \cdots + \mu_n' y_n = 0 \\ \mu_1' y_1' + \mu_2' y_2' + \cdots + \mu_n' y_n' = 0 \\ \qquad\vdots \\ \mu_1' y_1^{(n-2)} + \mu_2' y_2^{(n-2)} + \cdots + \mu_n' y_n^{(n-2)} = 0 \\ \mu_1' y_1^{(n-1)} + \mu_2' y_2^{(n-1)} + \cdots + \mu_n' y_n^{(n-1)} = g(t) \end{cases}.$$

Like in Problem 4 though, there is a formula for the particular solution in terms of the complementary solution with some integrals and Wronskians and stuff. Can you figure out what this formula is? Looking at patterns in the formula you get in Problem 4, and maybe working out the solution to the system of equations for a third-order linear differential equation will help you guess. Then after you think you have it, feel free to look up the general formula to check yours, and so you can say you've seen it once in your life.

6. Find the general solution to the differential equation

$$\ddot{y} - 2\dot{y} + y = \frac{e^t}{t^2+1}$$

using the method of variation of parameters. You may use the method from class, or the formulas from Problem 4 (but only if you've worked through Problem 4). This is Example 2 in Paul's Notes on Variation of Parameters, so feel free to check your solution with his.

7. Using whatever techniques you find that work, find the solution in each of the following:

   (a) Find the particular solution to the differential equation $y'' - 2y' + y = 4\cos(x)$ where $y(0) = \frac{\pi}{2} + 1$ and $y'\left(\frac{\pi}{2}\right) = e^{\frac{\pi}{2}}$.

   (b) Find the general solution to $y'' - y' - 2y = e^{3x}$.

   (c) Find the general solution to $\dot{y} - 5y = -xe^{5x} + x^2e^x$.

   (d) Find the particular solution to $y'' + 4y = \sin^2(2x)$ where $y(\pi) = 0$ and $y'(\pi) = 0$.

8. If you're into the physics side of things, and you find yourself asking, *"What sorts of physical situations do second-order linear differential equations model?"*, then you should at least skim through Paul's Notes on Mechanical Vibrations. The textbook situation for second-order linear differential equations is an object hanging from a spring, potentially with damping forces to consider. The differential equation you get is

$$m\ddot{u} + \gamma\dot{u} + ku = F(t)$$

where $u(t)$ is the function that gives the displacement of the object from equilibrium at time t, m is the mass of the object, γ is a coefficient associated to the damping force, k is the spring constant, and $F(t)$ is a catch-all for any other external forces acting on the object.

9. (OPTIONAL PRACTICE) Using whatever techniques you find that work, find the solution in each of the following. Solutions to these are posted at the bottom of the page.

   (a) Find the general solution to $y' - y = \sin(x) + \cos(2x)$.

   (b) Find the general solution to $y''' + y' = \sec(x)$.

   (c) Find the general solution to $\dddot{y} - 3\ddot{y} + 3\dot{y} - y = e^x + 1$.

Solutions to Problem 9.

1. $y = c_1 e^x - \frac{1}{2}\sin(x) - \frac{1}{2}\cos(x) + \frac{2}{5}\sin(2x) - \frac{1}{5}\cos(2x)$

2. $y = c_1 + c_2\cos(x) + c_3\sin(x) + \ln|\sec(x) + \tan(x)| - x\cos(x) + \sin(x)\ln|\cos(x)|$

3. $y = c_1 e^x + c_2 x e + c_3 x^2 e^x + \frac{1}{6}x^3 e^x - 1$