Homework 7

Ordinary Differential Equations

UCR Math-046-E01, Summer 2018

1. Compute the following Laplace transforms manually.

   (a) $\mathscr{L}\left\{x^2\right\}$

   (b) $\mathscr{L}\left\{te^{at}\right\}$ for some nonzero real number a.

   (c) $\mathscr{L}\{\sin(ax)\}$. You can either do it straight with sine, or use the fact that $\sin(ax) = \frac{1}{2i}(e^{iax} - e^{-iax})$.

   (d) (Challenge) $\mathscr{L}\left\{x^n\right\}$ for a natural number $n > 0$.

2. Compute the following Laplace transforms. You may use Paul's table of common Laplace transforms if you find that it helps.

   (a) $\mathscr{L}\{\cosh(3x)\}$

   (b) $\mathscr{L}\left\{\sqrt{t^3}\right\}$

   (c) $\mathscr{L}\left\{t^3\sinh(3t)\right\}$

   (d) $\mathscr{L}\{\sin(3t-2)\}$

   (e) $\mathscr{L}\left\{g(t)\right\}$ where $g(t) = \int_0^t \cos(42\mu)\,d\mu$.

   (f) $\mathscr{L}\left\{f(x)\right\}$ where $f(x) = \begin{cases} x & 0 \le x \le 2 \\ 2 & x > 2 \end{cases}$.

3. (The Gamma Function) You may have noticed the Gamma function Γ appear in the table of Laplace transforms we are using. You remember the factorial function $n! = n \times (n-1) \times \cdots \times 3 \times 2 \times 1$, right? The factorial function is only defined for positive integers, which is a bummer. The *Gamma function* is the continuous analogue of the factorial function, which is defined for all positive real numbers. The explicit definition of the Gamma function is

$$\Gamma(p+1) = \int_0^\infty e^{-t} t^p \,dt \qquad (\star)$$

   Notice that the input of Γ is offset by $+1$. Let's prove some properties of Γ that we'd expect it to have as the continuous analogue of the factorial function.

(a) Remember that $n! = n \times (n-1)!$. Can you demonstrate, using the definition ($\star$), for any positive real number p, that $\Gamma(p+1) = p\Gamma(p)$?

(b) Prove that $\Gamma(1) = 1$.

(c) Using the last two parts, show for any natural number n that $\Gamma(n+1) = n!$. I.e. with an offset of $+1$, Γ agrees with the factorial function in the positive integers.

(d) Here's a weird fact: $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$. Proving this fact is a little tough, so we won't do that here. Instead, using this fact, and using part (3a), what is $\Gamma\left(\frac{3}{2}\right)$? What is $\Gamma\left(\frac{7}{2}\right)$?

(e) With the fact that $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$ in mind, look back at items 3, 4, and 5 in Paul's table of common Laplace transforms, and see that you can derive items 3 and 5 from item 4 using what you now know about Γ.

4. (OPTIONAL PRACTICE) Finding inverse Laplace transforms often just comes down to either doing some gross partial fraction decomposition, or completing the square on some quadratics that won't factor nicely (you'll see). You'll have to do this a few times in the exercises ahead, so this exercise is really just to give you more practice, or a warm-up, or whatever you want. Find the inverse Laplace transforms of the following functions. A couple of these are Examples in Paul's Notes on Inverse Laplace Transforms.

   (a) $K(s) = \frac{1}{\sqrt{s}}$

   (b) $G(s) = \frac{2^{n+1} n!}{s^{n+1}}$

   (c) $F(s) = \frac{19}{s+2} - \frac{1}{3s-5} + \frac{7}{s^5}$

   (d) $G(s) = \frac{1-3s}{s^2+8s+21}$ (notice that the denominator doesn't factor nicely into linear factors, so you may want to complete the square down there)

   (e) $F(s) = \frac{s+4}{s^2+4s+8}$

5. Solve this initial value problem using the Laplace transform.

$$t\ddot{y} - t\dot{y} + y = 2 \quad \text{where} \quad y(0) = 2 \text{ and } \dot{y}(0) = -4$$

   Note that this is Example 2 in Paul's Notes on Solving IVPs with Non-constant Coefficients. Work through it yourself, and if you get stuck or are unsure of yourself at a step, consult Paul.

6. Solve the following initial value problems using the Laplace transform.

   (a) $y' - 5y = 0$ where $y(0) = 2$.

   (b) $\dot{y} - 5y = 0$ where $y(\pi) = 2$.

   (c) $y'' + 4y' + 8 = \sin(x)$ where $y(0) = 1$ and $y'(0) = 0$.

7. Compute the Laplace transform of the following function. Feel free to consult a table of common Laplace transforms.

$$f(t) = \begin{cases} t^5 & 0 \le t < 1 \\ e^{2t-2}\cosh(3(t-1)) + t^5 & t \ge 1 \end{cases}$$

8. Compute the Laplace transform of the following function. Feel free to consult a table of common Laplace transforms.

$$g(t) = \begin{cases} (t^2 + \pi^2)e^{2t} & 0 \le t < \pi \\ \cos(2t) + 2\pi t e^{2t} & t \ge \pi \end{cases}$$