ODEs - Week 2 - Monday (1)

x,y - linear substitutions

Given a differential equation of the form

$$y' = G(ax+by)$$

So y' is a function of some linear pattern $ax+by$, we can make the substitution

$$v = ax+by \qquad \& \qquad v' = a+by'$$
$$\left(y' \mapsto \frac{v'-a}{b}\right)$$

This is often helpful, and will usually result in a separable differential equation.

"Solve $y' - (4x-y+1)^2 = 0$ where $y(0) = ?$."

Notice y' is a function of the term $4x-y$

$$y' = G(4x-y) \text{ where } G(\_) = (\_+1)^2$$

So let's try the substitution

$$v = 4x-y \qquad v' = 4-y'$$

ODEs - Week 2 - Monday (2)

$$v = 4x - y \quad \& \quad \begin{array}{c} v' = 4 - y' \\ (y' \mapsto 4 - v') \end{array}$$

So our differential equation becomes

$$(4 - v') - (v + 1)^2 = 0$$

Since there are no explicit xs, only vs, this is certainly separable.

$$4 - v' - v^2 - 1 - 2v = 0$$

$$\Rightarrow \quad -v^2 - 2v + 3 = v'$$

$$\Rightarrow \quad \int dx = -\int \frac{1}{v^2 + 2v - 3}\, dv$$

$$\Rightarrow \quad x + C = -\int \frac{dv}{(v+3)(v-1)}$$

Oh shoot we've gotta do a partial-fraction decomposition on that right integrand :'o !

If
$$\frac{1}{(v+3)(v-1)} = \frac{A}{(v+3)} + \frac{B}{(v-1)} = \frac{Av - A + Bv + 3B}{(v+3)(v-1)}$$

for some real numbers A and B, then

$$\begin{cases} A + B = 0 \\ -A + 3B = 1 \end{cases}, \text{ so } B = \frac{1}{4} \text{ and } \cancel{A = \frac{3}{4}}. \quad A = -\frac{1}{4}$$

Then

$$-\int \frac{dv}{(v+3)(v-1)} = \cancel{+\frac{3}{4}}\,\frac{1}{4}\int \frac{1}{v+3}\,dv - \frac{1}{4}\int \frac{1}{v-1}\,dv$$

$$= \cancel{+\frac{3}{4}}\,\frac{1}{4}\ln(v+3) - \frac{1}{4}\ln(v-1)$$

$$= \frac{1}{4}\ln\left(\frac{(v+3)}{\cancel{(v+3)^3}(v-1)}\right)$$

Back to our original DE,

$$4x + C = \ln\left(\frac{(v+3)}{\cancel{(v+3)^3}(v-1)}\right)$$

$$Ce^{4x} = \frac{(4x-y+3)}{\cancel{(4x-y+3)}(4x-y-1)}$$

Not yet!

ODEs - Week 2 - Monday (4)

So we've got $Ce^{4x} = \frac{v+3}{v-1}$. Notice that there are two vs on the righthand side, which means there will be two ys when we sub back in the $v = 4x - y$. I only want one $y^{(v)}$, because that'll make it easier to solve for y later, so to get only one v,

$$\frac{v+3}{v-1} = \frac{v-1+1+3}{v-1} = \frac{(v-1)+4}{v-1} = 1 + \frac{4}{v-1}$$

So are DE has one v, we sub back, and solve for y,

$$Ce^{4x} = 1 + \frac{4}{v-1}$$

$$\Rightarrow \quad \frac{4}{Ce^{4x}-1} = v - 1$$

$$\Rightarrow \quad v = \frac{4}{Ce^{4x}-1} + 1$$

$$\Rightarrow \quad 4x - y = \text{''}$$

$$\Rightarrow \quad y = 4x - 1 - \frac{4}{Ce^{4x}-1}$$

This is the general solution. But this was an IVP!

ODEs - Week 2 - Monday (5)

Recall $y(0)=2$, so

$$(2) = 4(0) - 1 - \frac{1}{Ce^{4(0)} - 1}$$

$$\Rightarrow \quad 2 = -1 - \frac{1}{C-1}$$

$$\Rightarrow \quad \frac{1}{3} = -(C-1) \quad \Rightarrow \quad C = \frac{2}{3}$$

And our particular solution has a denominator of $\frac{2}{3}e^{4x} - 1$, so we can't have

$$\frac{2}{3}e^{4x} - 1 = 0$$

$$e^{4x} = \frac{3}{2}$$

$$x = \frac{1}{4}\ln\left(\frac{3}{2}\right),$$

So our particular solution is

$$y = 4x - 1 - \frac{3}{2e^{4x} - 3} \quad \text{on } \left(-\infty, \frac{1}{4}\ln\left(\frac{3}{2}\right)\right).$$

Wasn't that fun!

ODEs - Week 2 - Tuesday (or Monday)

<u>Aside</u> <u>into</u> <u>partial</u> <u>derivatives</u>

Back in 7A, 7B, 9A, 9B you dealt with a single variable x, and took derivatives wrt only x. If you've taken 10A though, you've already seen multivariable functions like

$$f(x,y) = x^2y + e^xy^3 - 3\ln(x+y),$$

where both x and y are variables (y is <u>not</u> a function of x right now).

You can take <u>partial</u> <u>derivatives</u> of our function f by choosing only one variable to ~~the~~ take the derivative wrt. The only variables you treat as constants.

$$\frac{\partial f}{\partial x} = f_x = 2xy + e^xy^3 - \frac{3}{x+y}$$

$$\frac{\partial f}{\partial y} = f_y = x^2 + 3e^xy^2 - \frac{3}{x+y}$$

Practice really quick:

$$f(x,y) = 3x^2y^4 + e^{xy} - 2x + 7y^2, \quad f_x? \quad f_y?$$

Also $\quad \dfrac{\partial}{\partial x}\left(f(x, y(x))\right) = f_x + f_y y'$

ODEs - Week 2 - Tuesday (1)

Exact Differential Equations

An *exact* differential equation is one of the form

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

OR

$$M(x,y) + N(x,y)\,y' = 0$$

if you prefer, such that there is some function $G(x,y)$ where $G_x = M$ and $G_y = N$, those being the partial derivatives of G. So the DE, recalling the multivariable chain rule, can be written as

$$G_x + G_y\, y' = 0$$

function of x

$$\Rightarrow \frac{\partial}{\partial x}\left(G(x,y) \right) = 0$$

$$\cancel{\Rightarrow \int \frac{\partial}{\partial x}\left(G(x,y) \right) = \int 0}$$

$$\Rightarrow G(x,y) = C$$

So this gives us an (implicit) solution to the DE.

ODEs - Week 2 - Tuesday (2)

So the game to solving exact DEs is to find G, but it would be nice to know if an equation was exact before we went off searching for G. Well if G does exist, recall that $G_{xy} = G_{yx}$, which means that $M_y = N_x$. This turns out to also be a sufficient condition for a DE to be exact.

The DE

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

is exact if and only if $M_y = N_x$.

" Solve the differential equation

$$2xy - 9x^2 + (2y + x^2 + 1)y' = 0 \quad . "$$

Rewrite this as $(2xy - 9x^2)\,dx + (2y^2 + x^2 + 1)\,dy = 0$;
maybe this is exact for $M = 2xy - 9x^2$, $N = 2y^2 + x^2 + 1$

$$M_y = 2x \quad \text{and} \quad N_x = 2x \, .$$

So it's exact! ☺

ODEs - Week 2 - Tuesday (3)

So there's some G such that

$$\begin{cases} G_x = M = 2xy - 9x^2 \\ G_y = N = 2y + x^2 + 1 \end{cases} \Longrightarrow \begin{cases} G = \underline{x^2y} \underline{-3x^3} + \underline{g(y)} \\ G = \underline{y^2} + \underline{x^2y} + \underline{y} + \underline{h(x)} \end{cases}$$

So $G = x^2y - 3x^2 + y^2 + y + C$

And since

$$M(x,y)\,dx + N(x,y)\,dy = 0,$$

$$\frac{\partial}{\partial x}\left(G(x,y)\right) = 0$$

$$\Rightarrow \quad G(x,y) = c$$

$$\Rightarrow \quad x^2y - 3x^2 + y^2 + y = c$$

is our solution.

How does everyone feel?

ODEs – Week 2 – Tuesday (4)

Another?

"Solve $y' = -\dfrac{x + \sin(y)}{x\cos(y) - 2y}$."

It's not homogeneous, and looks hard to separate, so let's pray that it's exact.

$$\underbrace{(x + \sin(y))}_{M}\, dx + \underbrace{(x\cos(y) - 2y)}_{N}\, dy = 0$$

$M_y = \cos(y)$ $\quad N_x = \cos(y)$ $\quad$ so it's exact! ☺

$$\begin{cases} G_x = x + \sin(y) \\ G_y = x\cos(y) - 2y \end{cases} \Rightarrow \begin{cases} G = \frac{1}{2}x^2 + x\sin(y) + g(y) \\ G = +x\sin(y) - y^2 + h(x) \end{cases}$$

So our solution G is $x\sin(y) + \frac{1}{2}x^2 - y^2 + C = 0$.

Another?

"$xe^{xy}\, dx + ye^{xy}\, dy = 0$" $\quad$ "$(2xy + x)\, dx + (x^2 + y)\, dy = 0$"

ODEs - Week 2 - Wednesday (1)

Linear first-order differential equations

First, let's introduce the word integrating factor. An integrating factor is some mysterious thing you multiply through a whole differential equation to make it easier to solve. ~~If you have~~

Now a linear first-order differential equation is one of the form

$$y' + p(x)y = q(x)$$

where $p(x)$ and $q(x)$ are continuous functions of x. Note that you might start with an equation like

$$r(x)y' + p(x)y = q(x)$$

but you can, after worrying about where $r(x)=0$, write this as

$$y' + \frac{p(x)}{r(x)}y = \frac{q(x)}{r(x)}$$

where those functions are continuous everywhere $r(x) \neq 0$.

ODEs - Week 2 - Wednesday (2)

We can solve a <u>first-order</u> <u>linear</u> DE by making a clever choice of integrating factor.

First, an analogy. Does everyone remember how to "complete the square" of a quadratic?

$$y^2 + 6y = 7$$

$$y^2 + 6y + 9 - 9 = 7 \quad \text{(add something clever)}$$

$$(y+3)^2 = 16 \quad \checkmark \quad \text{(unexpand)}$$

We're going to do a similar trick here. We're gonna multiply through by a clever integrating factor (add something clever) and then undo the product rule (unexpand).

See if you have a function $\mu(x)$, the derivative

$$\frac{\partial}{\partial x}(\mu y) = \mu y' + \mu' y.$$

looks kinda first-order linear.

ODEs - Week 2 - Wednesday (3)

So the dream is for some function $\mu(x)$ such that when we multiply it through our linear first-order DE we have $\mu' = \mu p$, because then

$$y' + py = q$$

$$\mu y' + \mu p y = \mu q$$

$$\mu y' + \mu' y = \mu q$$

$$(\mu y)' = \mu q$$

and then we have a single y and can just integrate both sides wrt x to solve the DE. But how can we find such a fuction μ when $\mu' = \mu p$? By being smart

$$\frac{\partial}{\partial \mu} \ln(\mu)\,? = \frac{\mu'}{\mu} = p$$

integrating both sides

$$\ln(\mu) = \int p \, \partial x$$

$$\mu = e^{\int p \, \partial x}$$

So this is our integrating factor!

ODEs - Week 2 - Wednesday (4)

How does everyone feel? Let's try it out.

"Solve $xy' + 2y = x^2 - x + 1$ where $y(1) = \frac{1}{2}$."

First we're gotta make it linear, so let's divide through by x, which may affect the domain of our solution.

$$y' + \frac{2}{x}y = \left(x - 1 + \frac{1}{x}\right) \qquad (x \neq 0)$$

In this form $p(x) = \frac{2}{x}$, so our clever integrating factor is $\mu(x) = e^{\int p(x)\,dx} = e^{\int \frac{2}{x}\,dx} = e^{2\ln(x)} = x^2$.

Multiplying through

$$\underbrace{x^2 y' + 2xy} = x^2\left(x - 1 + \frac{1}{x}\right)$$

$$(x^2 y)' = x^3 - x^2 + x$$

$$\int (x^2 y)'\,dx = \int x^3 - x^2 + x\,dx$$

$$x^2 y = \frac{1}{4}x^4 - \frac{1}{3}x^3 + \frac{1}{2}x + C$$

$$\text{eh}\begin{cases} y = \frac{1}{4}x^2 - \frac{1}{3}x + \frac{1}{2} + \frac{C}{x^2} & (x \neq 0) \\ y = \frac{1}{12x^2}\left(3x^4 - 4x^3 + 6x^2 + C\right) \end{cases}$$

ODEs - Week 2 - Wednesday (5)

Then our initial condition is that $y(1) = \frac{1}{2}$, so

$$\frac{1}{2} = \frac{1}{12}(3 - 4 + 6 + c)$$

$$6 = (5 + c), \quad \text{so} \quad C = 1$$

Then our solution is $y = \dfrac{3x^4 - 4x^3 + 6x^2 + 1}{12x^2}$ on $(0, \infty)$.

How does everyone feel?

$$t\dot{y} - 2y = t^5 \sin(2t) - t^3 + 4t^4$$

Hard though..

And same integrating factor..."

Put it on final

Better

$$2y' - y = 4\sin(3t)$$

ODEs - Week 2 - ~~Wednesday~~ Thursday (1)

Bernoulli Differential Equations

Early 1700s, named after Jacob Bernoulli, $e^{i\pi} = -1$
brother of Johann Bernoulli and uncle of Daniel.
Sided with Leibniz. "Discovered" $e = 2.7182818284$

A <u>Bernoulli differential equation</u> has the form

$$y' + p(x)y = q(x)y^n.$$

Looks nearly linear, but for that y^n factor. Again you might be handed something with a factor of $r(x)$ as a coefficient on y', but you can just divide through by $r(x)$ and exclude everywhere $r(x)=0$ from your domain.

We can do a quick substitution to make this DE linear first-order.

First rearrange it a bit

$$y^{-n}y' + p(x)y^{1-n} = q(x)$$

and then make the substitution

$$y^{1-n} \mapsto v \qquad \& \qquad (1-n)y^{-n}y' \mapsto v'$$
$$y^{-n}y' \mapsto v'\frac{1}{1-n}$$

which gets us

$$\frac{1}{1-n}v' + p(x)v = q(x)$$

$$\Rightarrow \quad v' + (1-n)p(x)v = (1-n)q(x)$$

which is linear.

" Solve $y' + \frac{4}{x}y = x^3y^2$ where $\begin{matrix} x>0 \\ y(2)=-1 \end{matrix}$. "

ODEs – Week 2 – Thursday (3)

$$y^{-2}y' + \frac{4}{x}y^{1-2} = x^3$$

$$\overset{1-2}{y^{-1}} \mapsto v \qquad \& \qquad -1y^{-2}y' \mapsto v'$$

$$y^{-2}y' \mapsto -1v'$$

$$-1v' + \frac{4}{x}v = x^3$$

$$v' - \frac{4}{x}v = -x^3$$

The sign is important!

Multiply through by $e^{-\int \frac{4}{x}dx} = e^{-4\ln(x)} = x^{-4}$ since its linear

$$x^{-4}v' - 4x^{-3}v = -x^{-1}$$

$$\int \frac{\partial}{\partial x}\left(x^{-4}v\right)dx = \int -\frac{1}{x}\,dx$$

$$x^{-4}v = -\ln(x) + C$$

$$v = x^4\ln\left(\frac{c}{x}\right)$$

$$\frac{1}{y} = x^4\ln\left(\frac{c}{x}\right)$$

$$y = \frac{1}{x^4\ln\left(\frac{c}{x}\right)}$$

ODEs – Week 2 – Thursday (4)

In particular, when $y(2) = -1$, $x > 0$

$$-1 = \frac{1}{(2)^4 \ln\left(\frac{c}{2}\right)}$$

$$-1 = 16 \ln\left(\frac{c}{2}\right)$$

$$e^{-1/16} = \cancel{\ln}\left(\frac{c}{2}\right)$$

$$C = 2e^{-1/16}$$

Then notice that $\ln\left(\frac{2e^{-1/16}}{x}\right) = \ln\left(2e^{-1/16}\right) - \ln(x)$

$$= \ln(2) - \frac{1}{16}\ln(e) - \ln(x)$$

$$= \ln\left(\frac{2}{x}\right) - 1/16$$

So our particular solution is

$$(2e^{-1/16}, \infty)$$

$$y = \frac{1}{x^4\left(\ln\left(\frac{2}{x}\right) - 1/16\right)} \quad \text{on} \quad \cancel{(0, 2e^{-1/16})}$$

How y'all feel!

$\frac{2}{x} > 0$ and $\ln\left(\frac{2}{x}\right) - 1/16 \neq 0$

good for $x > 0$

$$\frac{2}{x} \neq e^{1/16}$$

$$2e^{-1/16} \neq x$$

Domain breaks at

0 and $2e^{-1/16}$

> 1

< 2 tho

ODEs - Week 2 - Thursday (5)

Try on "Solve $y' + xy = xy^2$."

Do I have the courage to ask...
next week, should we do
- Applications?
- Second-Order DEs?