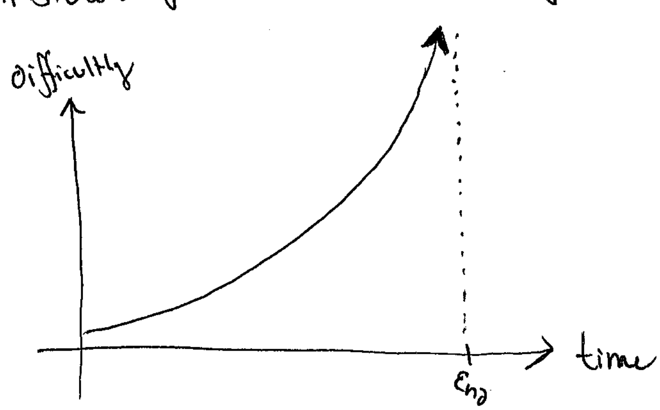
ODEs - Week 7 - Tuesday (1)

This week, we're going to start a crash course on linear algebra and systems of DEs. So, two things to keep in mind:

1) We're going to go fast, so ask any question you can think of and don't feel too bad if you're not solid on what I'm talking about. Seeing this stuff even briefly now will help you understand it better when if it comes up later in another class.

2) If there's anything on the final about this stuff, it'll be pretty easy. Like, just making sure you're thinking and being attentive in lecture. Outside of class, while studying for the final, you should be focusing your time on the practice midterm and post quizzes instead of the material of this week.

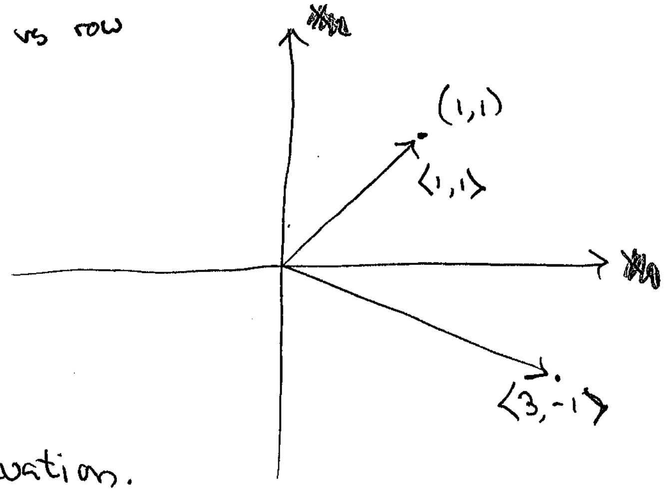
Difficulty of a course over time:



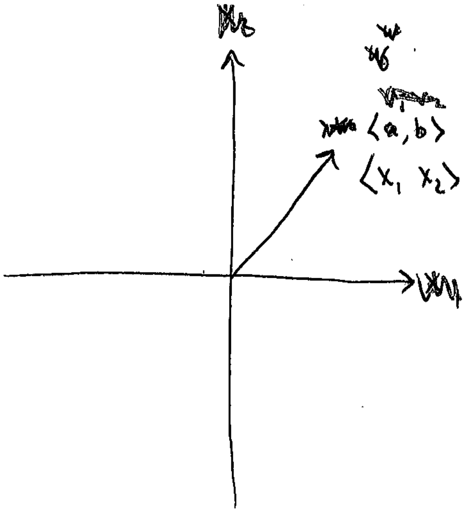
ODEs - Week 7 - Tuesday (2)

Everything we're going to do now could happen in n-dimensional space, but we'll only look at $n = 2, 3$ dimensions b/c those are easier in terms of calculations and drawings.

You can think of a <u>vector</u> either as a point in space or an (column vs row) an arrow pointing from the origin to that point. Whichever is convenient for your situation.



This place where vectors live is called a <u>vector space</u>. We'll work over $\mathbb{R}$, and the picture above is a 2-dimensional vector space over $\mathbb{R}$. A <u>linear map</u> between vector spaces is a function where the coordinate functions are linear in the inputs

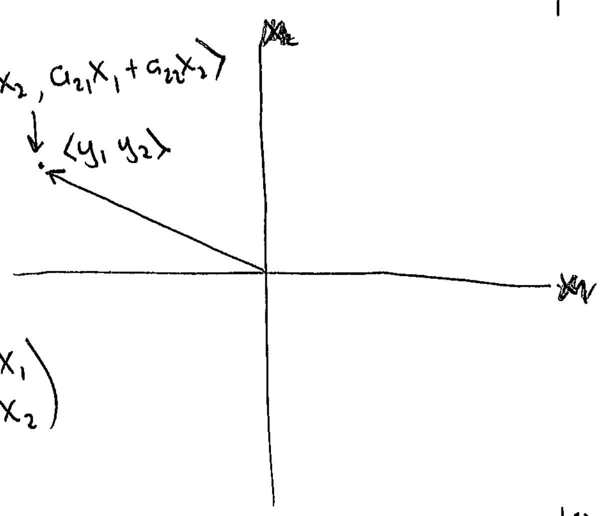


$$\varphi(\langle \overset{x_1}{a}, \overset{x_2}{b} \rangle) = \langle a_{11}x_1 + a_{12}x_2,\ a_{21}x_1 + a_{22}x_2 \rangle \downarrow \langle y_1\ y_2 \rangle$$

$$\begin{cases} y_1 = a_{11}x_1 + a_{12}x_2 \\ y_2 = a_{21}x_1 + a_{22}x_2 \end{cases}$$

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\vec{y} = A\vec{x}$$



ODEs - Week 7 - Tuesday (3)

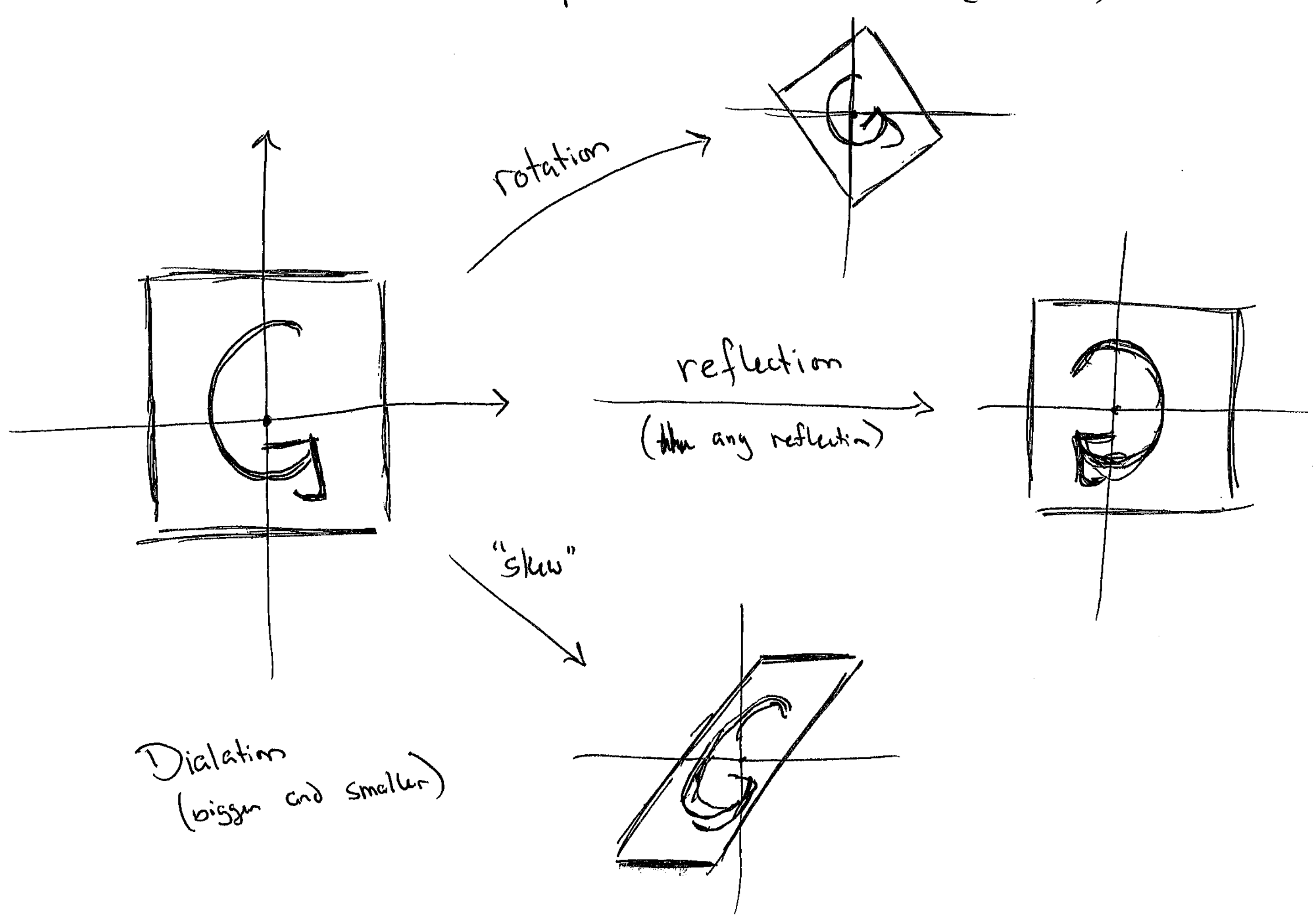
For these ~~sys~~ linear maps, you can write them as a system of linear equations, which in turn you can write as an equation with matrices and vectors. For example

$$\varphi(\langle x_1, x_2\rangle) = \langle \underbrace{x_1 + 2x_2}_{y_1}, \underbrace{7x_1 - 3x_2}_{y_2} \rangle$$

$$\begin{cases} y_1 = x_1 + 2x_2 \\ y_2 = 7x_1 - 3x_2 \end{cases} \equiv \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \overset{A}{\begin{pmatrix} 1 & 2 \\ 7 & -3 \end{pmatrix}} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\equiv \vec{y} = A\vec{x} \quad \checkmark$$

Now what do "linear" maps look like? (in $\mathbb{R}^2$)



ODEs - Week 7 - Tuesday (4)

Now let's talk about eigenvectors and eigenvalues.

An eigenvector ~~is vector~~ of a linear transformation are vectors that point in the same direction after you apply the transformation. They might get scaled though, and the amount that an eigenvector gets scaled by under a transformation is it's eigenvalue. (Look at examples from previous page).

Now what does this mean algebraically? $\vec{v}$ is an eigenvector if

$$A\vec{v} = \lambda\vec{v} \qquad \begin{pmatrix} \vdots & \ddots & \\ \vdots & & \ddots \end{pmatrix}\begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} = \lambda\begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$$

This is nifty, but given some linear transformation A, how can we hope to find the eigenvectors/eigenvalues? Some algebra magic and the determinant ~~map~~ function.

$$A\vec{v} = \lambda\vec{v}$$

$$(A\vec{v} - \lambda\vec{v}) = 0 \qquad I = \begin{pmatrix} 1 & \cdots & 0 \\ \vdots & 1 & \vdots \\ 0 & \cdots & 1 \end{pmatrix}$$

$$(A - \lambda I)\vec{v} = 0$$

$$\underbrace{\det(A - \lambda I)}_{} = 0 \quad \checkmark$$

this'll be a polynomial in λ that we call the characterist equation.

ODEs - Week 7 - Tuesday (5)

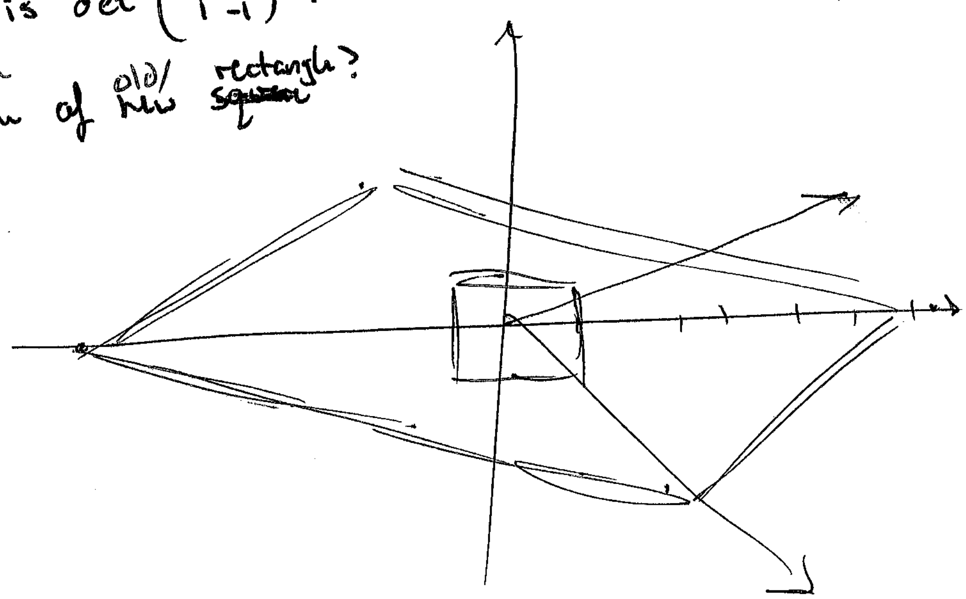
So the eigenvalues of a transformation given by A are the roots of the polynomial given by $\det(A-\lambda I)$.

"Consider the linear map given by the matrix $\begin{pmatrix} 2 & 4 \\ 1 & -1 \end{pmatrix}$. * What is the image of $\boxed{R}$ look like under this map? *(first) where does this map send the vector $\begin{pmatrix} 5 \\ -7 \end{pmatrix}$? What are the eigenvectors and eigenvalues of this map?"

What is $\det\begin{pmatrix} 2 & 4 \\ 1 & -1 \end{pmatrix}$?

~~Volume~~ Area of old/ ~~new~~ rectangle? ~~square~~

Important: draw this to scale



"Do it again for $\begin{pmatrix} 1 & 3 \\ 4 & 5 \end{pmatrix}$."

ODEs - week 7 - Wednesday (1)

Systems of differential equations (at least linear homogeneous ones at first). Previous to today, there would only be a single function we'd look at at a time. Now were going to look at a system of DEs of multiple (related functions). A solution to a system of DEs is a set of functions that satisfies each equation in the system simultaneously. So say:

$$\begin{cases} y_1' = y_1 + 2y_2 \\ y_2' = 3y_1 + 2y_2 \end{cases}$$

this is a system of two linear homogeneous DEs, and a solution to this system would be any $\{y_1, y_2\}$ that satisfies those two equations. We call such a system where y_1 depends on y_2 and vice-versa y_2 depends on y_1 a coupled system.

If time permits, we'll model some things with these systems, like two tanks entering into each other, or predator-prey models where the predation rate isn't constant.

ODEs - Week 7 - ~~Thursday~~ Wednesday (2)

First, just some abstract ODEs though.

We can write (by changing variables) any $n^{\underline{th}}$ order linear DE as a system of n linear DEs. For example,

"Write $2\ddot{\gamma} - 5\dot{\gamma} + \gamma = 0$ when $\gamma(3) = 6$ and $\dot{\gamma}(3) = -1$ as an equivalent system of DEs."

The idea is to just let each derivative $\gamma, \dot{\gamma}, \ddot{\gamma}, \ldots$ be its own function. Let

$$\vec{y}' = \begin{pmatrix} 0 & 1 \\ -1/2 & 5/2 \end{pmatrix} \vec{y}$$

$$y_1 = \gamma \qquad y_2 = \dot{\gamma}, \quad \text{so} \quad \dot{y}_1 = y_2 = \dot{\gamma}, \quad \dot{y}_2 = \ddot{\gamma}$$

$$\begin{cases} \dot{y}_1 = 0 + y_2 \\ \dot{y}_2 = -\frac{1}{2}y_1 + \frac{5}{2}y_2 \end{cases}$$

$$y_1(3) = 6 \qquad y_2(3) = -1.$$

"How do you write the DE $\ddddot{\gamma} + 3\dddot{\gamma} - \sin(t)\ddot{\gamma} + 8\dot{\gamma} = t^2$ as a system of first-order DEs?"

$$\begin{cases} y_1 = \gamma \\ y_2 = \dot{\gamma} \\ y_3 = \ddot{\gamma} \\ y_4 = \dddot{\gamma} \end{cases} \Rightarrow \begin{cases} y_1' = y_2 \\ y_2' = y_3 \\ y_3' = y_4 \\ y_4' = -3y_3 + \sin(t)y_2 - 8y_1 + t^2 \end{cases}$$

$$\vec{y}' = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -8 & \sin(t) & -3 & 0 \end{pmatrix} \vec{y} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ t^2 \end{pmatrix}$$

linear
<u>not</u>-homogeneous

ODEs - Week 7 - Wednesday (3)

Now how do we solve such a system? We'll focus only on linear homogeneous systems first.

$$\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 \\ -\frac{1}{2} & \frac{5}{2} \end{pmatrix}}_{A} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \quad \text{OR} \quad \vec{y}' = A\vec{y}$$

That looks very similar to a case we've seen many times in 1-dimension: $y' = ay$. (where $y = c_1 e^{at}$.)

We should let this guide our guess for the solution to $\vec{y}' = A\vec{y}$. I'll bet $\vec{y} = \vec{\eta} e^{at}$ is a solution, where

$$\vec{y} = \vec{\eta} e^{at} = \begin{pmatrix} \eta_1 \\ \vdots \\ \eta_n \end{pmatrix} e^{at} = \begin{pmatrix} \eta_1 e^{at} \\ \vdots \\ \eta_n e^{at} \end{pmatrix}$$

Let's check it!

$$\vec{y}' = \left(\vec{\eta} e^{at}\right)' = a\vec{\eta} e^{at}, \quad \text{and we'd need}$$

$$a\vec{\eta} e^{at} = A\left(\vec{\eta} e^{at}\right)$$

$$a\vec{y} = A\vec{y}$$

Remember!? This means a is an eigenvalue for A!

ODEs - Week 7 - Wednesday (4)

Writing the whole thing out, since $e^{at} \neq 0$ ever,

$$a\vec{\eta}e^{at} = A\vec{\eta}e^{at} \Rightarrow (aI_n - A)\vec{\eta}e^{at} = 0$$
$$\Rightarrow (aI_n - A)\vec{\eta} = 0$$

So yeah, a is an eigenvalue, and $\vec{\eta}$ is the corresponding eigenvector. Then, to solve the system, we just need to find these eigenvectors and eigenvalues. And like before, we can get <u>all</u> the solutions this way.

Theorem

- If $\vec{y}_1$ and $\vec{y}_2$ are solutions to a system of homogeneous linear equations, then so is
$$c_1\vec{y}_1 + c_2\vec{y}_2.$$

- Given a homogenous linear system of n equations and n unknowns, if $\vec{y}_1, \dots, \vec{y}_n$ are each solutions such that

$$\det W = \det(\vec{y}_1, \dots, \vec{y}_n) = \det\begin{pmatrix} y_{11} & \cdots & y_{n1} \\ \vdots & \ddots & \vdots \\ y_{1n} & \cdots & y_{nn} \end{pmatrix} \neq 0$$

then the ~~solutions~~ general solution is $y = c_1\vec{y}_1 + \dots + c_n\vec{y}_n$.

ODEs - Week 7 - ~~Thursday~~ Wednesday (5)

"Solve $\vec{y}' = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix} \vec{y}$ where $\vec{y}(0) = \begin{pmatrix} 0 \\ -4 \end{pmatrix}$."

We just found that solutions look like $\vec{n} e^{at}$ where a is an eigenvalue of $\begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}$, so lets find those and their corresponding eigenvectors.

$$\det \begin{pmatrix} 1-\lambda & 2 \\ 3 & 2-\lambda \end{pmatrix} = (1-\lambda)(2-\lambda) - 6 = \lambda^2 - 3\lambda - 4 = (\lambda-4)(\lambda+1)$$

Then for our eigenvalue 4, we get the eigenvector $n_4 = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ like

$$An_4 = 4n_4 \Rightarrow \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 4 \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \Rightarrow \begin{cases} v_1 + 2v_2 = 4v_1 \\ 3v_1 + 2v_2 = 4v_2 \end{cases} \Rightarrow n_4 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

Then for the eigenvalue -1

$$An_{-1} = -1n_{-1} \quad \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = -1 \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \Rightarrow \begin{cases} v_1 + 2v_2 = -v_1 \\ 3v_1 + 2v_2 = -v_2 \end{cases} \Rightarrow n_{-1} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

So our general solution looks like

$$\vec{y} = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 2 \\ 3 \end{pmatrix} e^{4t}$$

Then to find the values of c_1 and c_2,

$$\vec{y}(0) = c_1\begin{pmatrix}1\\-1\end{pmatrix}e^{-(0)} + c_2\begin{pmatrix}2\\3\end{pmatrix}e^{4(0)}$$

$$\begin{pmatrix}0\\-4\end{pmatrix} = c_1\begin{pmatrix}1\\-1\end{pmatrix} + c_2\begin{pmatrix}2\\3\end{pmatrix}$$

$$\begin{cases} 0 = c_1 + 2c_2 \\ -4 = -c_1 + 3c_2 \end{cases} \Rightarrow \quad c_2 = -\frac{4}{5} \quad c_1 = \frac{8}{5}$$

So our particular solution is $\vec{y} = \begin{pmatrix}8/5\\-8/5\end{pmatrix}e^{-t} + \begin{pmatrix}-8/5\\-12/5\end{pmatrix}e^{4t}$. ✓

Or, as a system

$$\begin{cases} y_1 = \frac{8}{5}e^{-t} - \frac{8}{5}e^{4t} \\ y_2 = -\frac{8}{5}e^{-t} - \frac{12}{5}e^{4t} \end{cases}$$

"Find the general solution to $2\ddot{\gamma} + 5\dot{\gamma} - 3\gamma = 0$ by considering it as a system of DEs."

As a system this thing looks like

$$\begin{matrix} y_1 = \gamma \\ y_2 = \dot{\gamma} \end{matrix} \Rightarrow \begin{cases} \dot{y}_1 = y_2 \\ \dot{y}_2 = \frac{3}{2}y_1 - \frac{5}{2}y_2 \end{cases}$$

$$\Rightarrow \vec{y}\,' = \begin{pmatrix}0 & 1\\ 3/2 & -5/2\end{pmatrix}\vec{y}$$

ODEs – Week 7 – Thursday (7)

So to find the eigenvalues, we find the roots of

$$0 = \det\begin{pmatrix} 0-\lambda & 1 \\ 3/2 & -5/2-\lambda \end{pmatrix} = +\lambda\left(5/2+\lambda\right) - 3/2$$

$$= 5/2\,\lambda + \lambda^2 - 3/2$$

$$= 2\lambda^2 + 5\lambda - 3$$

Oh snap! It's the same "characteristic equation" that we've encountered before!

That's all, I don't wanna finish this one.

Notice that ~~this~~ ~~poly~~ poly from the last example had real distinct roots, but you should remember, after seeing the similarity to how we've solved these before, that there are two other cases:

- Complex ~~roots~~ eigenvalues
- Repeated ~~roots~~ eigenvalues

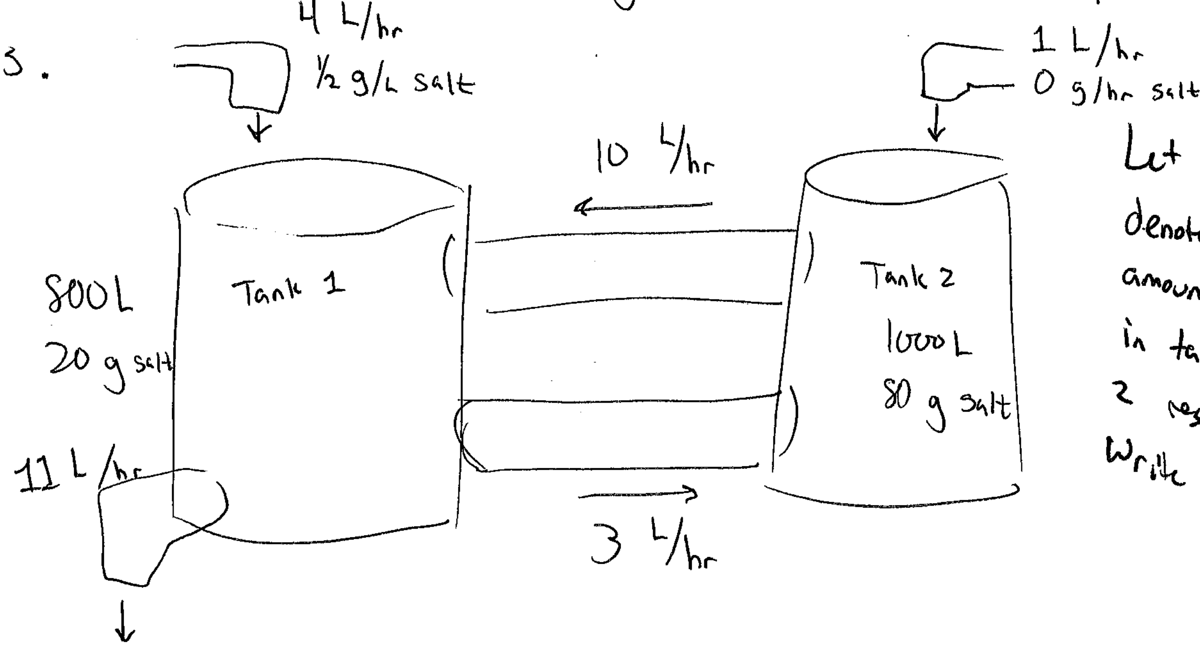
The way we deal with these cases is similar to before, and not sufficiently interesting to talk about.

Same with nonhomogeneous systems $\vec{y} = A\vec{y} + \sum f_i \vec{c}_i$

ODEs - Week 7 - Thursday (1)

More Modelling:

Now that we're talking about systems of DEs, we can model more complicated situations. Before, our models had a single function f, and it had inputs affecting it that didn't depend on f. But what if there were feedback to the input from f? Relating to our good 'ol tanks of water, what if the mixing tank emptied back into the first tank too? Or when we were modeling populations, suppose our animals were prey to some predator, and the populations of the predator and prey depended on each other? Let's look at just a couple examples of this.



Let Q_1, Q_2 denote the amount of salt in tank 1 and 2 respectively. Write the IVP

ODEs - Week 7 - Thursday (3)

Now let's talk about predator-prey models in blazing generality. In our model we'll have two populations P_1 and P_2 of the predator and prey respectively. Let's make some reasonable assumptions

- The population of prey <u>increases</u> at a rate proportional to its population in the absense of predators.
- The population of <u>predators</u> decreases at a rate propotial to its population in the absensense of prey.
- The number of times a prey and predator meet will be proportional to the products of their populations, and furthermore each encounter will decrease the prey population by 1 and increase the predator population.

Say that the ~~prey~~ predator ~~increases~~ decrease at a rate of a times its pop, and prey b. Then say a predator must eat n prey before it has 1 offspring

$$\begin{cases} \dot{P}_1 = -aP_1 + \frac{1}{n}P_1P_2 \\ \dot{P}_2 = bP_2 - P_1P_2 \end{cases} \Rightarrow \begin{cases} \dot{P}_1 = P_1\left(\frac{1}{n}P_2 - a\right) \\ \dot{P}_2 = P_2(b - P_1) \end{cases}$$

not even linear!