# Pop Quiz

Ordinary Differential Equations

UCR Math-046-E01, Summer 2018

1. With the initial conditions that $y(0) = 0$, $y'(0) = 7$, and $y''(0) = 20$, find the particular solution to the differential equation

$$y''' - 4y'' + 4y' = 0\,.$$

This is a 3rd-order linear homogeneous differential equation with constant coefficients, so we can solve it by looking at the roots of the corresponding characteristic equation.

$$\begin{aligned} \lambda^3 - 4\lambda^2 + 4\lambda &= 0 \\ \lambda(\lambda - 2)^2 &= 0\,. \end{aligned}$$

It has a root of $\lambda = 0$ and a root of $\lambda = 2$ of multiplicity two, so our general solution has the form

$$y(t) = c_1 + c_2 e^{2t} + c_3 t e^{2t}\,.$$

We know this must be the general solution since it is the sum of three linearly independent solutions, matching the order of the original differential equation. To solve for $c_1$, $c_2$, and $c_3$ we must calculate $y'$ and $y''$, use the initial conditions to set up a system of equations.

$$\begin{cases} y(t) = c_1 + c_2 e^{2t} + c_3 t e^{2t} \\ y'(t) = 2c_2 e^{2t} + c_3\left(2te^{2t} + e^{2t}\right) \\ y''(t) = 4c_2 e^{2t} + c_3\left(4te^{2t} + 4e^{2t}\right) \end{cases} \implies \begin{cases} 0 = c_1 + c_2 \\ 7 = 2c_2 + c_3 \\ 20 = 4c_2 + 4c_3 \end{cases}\,.$$

We can solve the system to find that $c_1 = -2$, $c_2 = 2$, and $c_3 = 3$. So our particular solution is

$$y(t) = 2e^{2t} + 3te^{2t} - 2\,.$$

There is another question on the back.

2. A 50 gallon tank initially contains 10 gallons of salt-water with a total of 10 lbs of salt dissolved in it. At $t = 0$ a salt-water solution containing 1 lb of salt per gallon is poured into the tank at a rate of 4 gal/min, while the well-stirred mixture leaves the tank at a rate of 2 gal/min. How much salt is in the tank at the moment that it overflows?

   Let $Q(t)$ denote the amount of salt in the tank at time $t$. We need to find $Q(t)$ at the time when the tank overflows. Since the tank is gaining 2 gal of water per minute, the tank will overflow at the time $t$ when $50 = 10+2t$, so when $t = 20$. We need to find $Q(20)$.

   To set up an initial value problem that models this situation, recall that the rate at which $Q$ is changing is the rate at which water is flowing into (respectively out of) the tank times the concentration of salt in that water. So our differential equation is

$$\dot{Q} = (4)(1) - (2)\left(\frac{Q}{10+2t}\right)$$

$$\Longrightarrow \quad \dot{Q} + \left(\frac{Q}{5+t}\right) = 4 \quad \text{where} \quad Q(0) = 10\,\text{lbs of salt}.$$

   This is a first-order linear differential equation, so we can solve it by multiplying through by the integrating factor $e^{\int p(x)\,dt}$ where $p(x) = \frac{1}{5+t}$. Doing this gives us

$$(5+t)\dot{Q} + (5+t)\left(\frac{Q}{5+t}\right) = 4(5+t)$$

$$\int \frac{d}{dt}(Q)(5+t)\,dt = \int (20+4t)\,dt$$

$$Q(t) = \frac{C+20t+2t^2}{5+t}$$

$$Q(t) = \frac{50+20t+2t^2}{5+t} \quad \text{since } Q(0) = 10.$$

   Then the amount of salt in the tank when it overflows is

$$Q(20) = \frac{50+20(20)+2(20)^2}{5+(20)} = \frac{50+400+800}{25} = 50\,\text{lbs}.$$