Pop Quiz

Ordinary Differential Equations

UCR Math-046-E01, Summer 2018

1. Suppose that $\sin(x)$ and x are complementary solutions, the solutions to the corresponding homogeneous differential equation, to

$$\big(\sin(x) - x\cos(x)\big)\,\ddot{y} - x\sin(x)\,\dot{y} + \sin(x)\,y = \tan(x)\big(\sin(x) - x\cos(x)\big).$$

Using the method *variation of parameters*, what is the particular solution to this equation? Feel free to leave your answer in terms of one or more integrals.

To use the method of *variation of parameters* we need the leading coefficient on $\ddot{y}$ to be 1, so we need to put this differential equation in the proper form by dividing everything through by $\sin(x) - x\cos(x)$. So the right-hand side becomes just $\tan(x) = g(x)$. Let $y_1 = \sin(x)$ and $y_2 = x$. We could recall that there's a formula for the particular solution

$$Y_p = -y_1 \int \frac{g\,y_2}{\mathrm{W}(y_1, y_2)}\,\mathrm{d}x + y_2 \int \frac{g\,y_1}{\mathrm{W}(y_1, y_2)}\,\mathrm{d}x \qquad (\star)$$

and just plug-and-play, but let's work it out a little.

We guess that our particular solution is of the form $Y_p = \mu_1 y_1 + \mu_2 y_2$, and we need to solve the system

$$\begin{cases} \dot{\mu}_1 y_1 + \dot{\mu}_2 y_2 = 0 \\ \dot{\mu}_1 \dot{y}_1 + \dot{\mu}_2 \dot{y}_2 = g(x) \end{cases} \quad = \quad \begin{cases} \dot{\mu}_1 \sin(x) + \dot{\mu}_2 x = 0 \\ \dot{\mu}_1 \cos(x) + \dot{\mu}_2 = \tan(x) \end{cases}.$$

Doing this, I get that

$$\dot{\mu}_1 = \frac{x\tan(x)}{x\cos(x) - \sin(x)} \qquad \dot{\mu}_2 = \tan(x) - \frac{x\sin(x)}{x\cos(x) - \sin(x)},$$

and so looking back at our guess, our particular solution must be

$$Y_p = \sin(x) \int \frac{x\tan(x)}{x\cos(x) - \sin(x)}\,\mathrm{d}x + x \int \left(\tan(x) - \frac{x\sin(x)}{x\cos(x) - \sin(x)}\right)\mathrm{d}x.$$

While this doesn't *immediately* look like what you get by using $(\star)$, they can be rearranged to look the same.

There is another question on the back.

2. Compute the Laplace transform of the following function. I will write some (possibly) helpful Laplace transforms on the board, and you may consult your own table of common Laplace transforms.

$$g(t) = \begin{cases} t^2 & 0 \le t < 2 \\ e^{3t-6} + 4t - 4 & 2 < t \end{cases}$$

First we can rewrite g as a single expression using a Heaviside function

$$g(t) = t^2 + u_2(t)\left(e^{3t-6} + 4t - 4 - t^2\right).$$

Then we can simply take the Laplace transform from here:

$$\begin{aligned} \mathscr{L}\{g(t)\} = &\ \mathscr{L}\left\{t^2 + u_2(t)\left(e^{3t-6} + 4t - 4 - t^2\right)\right\} \\ = &\ \mathscr{L}\left\{t^2\right\} + \mathscr{L}\left\{u_2(t)\left(e^{3t-6} + 4t - 4 - t^2\right)\right\} \end{aligned}$$

By recalling the three common Laplace transforms

$$\mathscr{L}\{t^n\} = \frac{n!}{s^{n+1}} \qquad \mathscr{L}\{u_c(t) f(t-c)\} = e^{-cs}\mathscr{L}\{f(t)\} \qquad \mathscr{L}\{e^{at}\} = \frac{1}{s-a},$$

we can continue with the calculation:

$$\begin{aligned} \mathscr{L}\{g(t)\} = &\ \mathscr{L}\left\{t^2\right\} + \mathscr{L}\left\{u_2(t)\left(e^{3t-6} + 4t - 4 - t^2\right)\right\} \\ = &\ \mathscr{L}\left\{t^2\right\} + \mathscr{L}\left\{u_2(t)\left(e^{3(t-2)} - (t-2)^2\right)\right\} \\ = &\ \frac{2}{s^3} + e^{-2s}\mathscr{L}\left\{e^{3t} - t^2\right\} \\ = &\ \frac{2}{s^3} + e^{-2s}\left(\mathscr{L}\left\{e^{3t}\right\} - \mathscr{L}\left\{t^2\right\}\right) \\ = &\ \frac{2}{s^3} + e^{-2s}\left(\frac{1}{s-3} - \frac{2}{s^3}\right) \\ = &\ \frac{e^{-2s}}{s-3} + \frac{2 - 2e^{-2s}}{s^3} \\ = &\ \frac{s^3 + 2(s-3)\left(e^{2s} - 1\right)}{s^3(s-3)e^{2s}} \qquad \text{(if one feels compelled to)} \end{aligned}$$