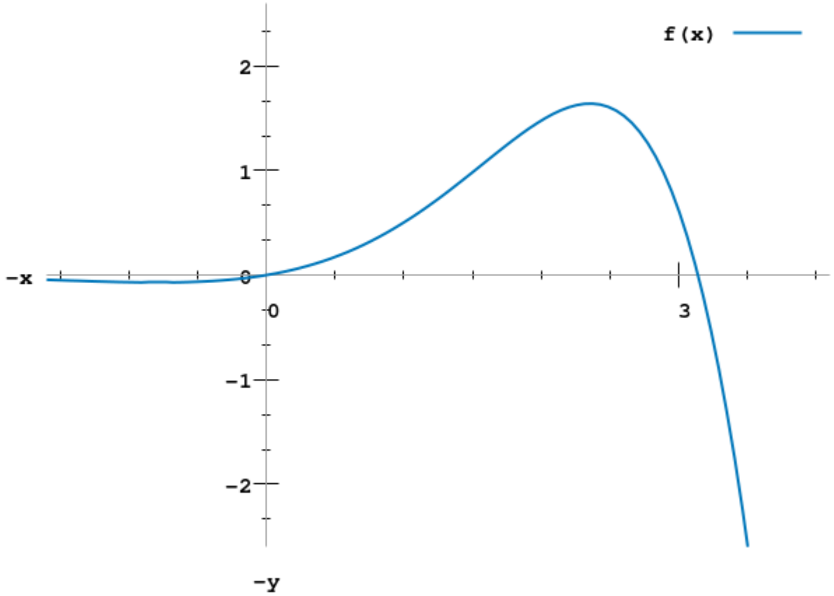
Pop Quiz 3

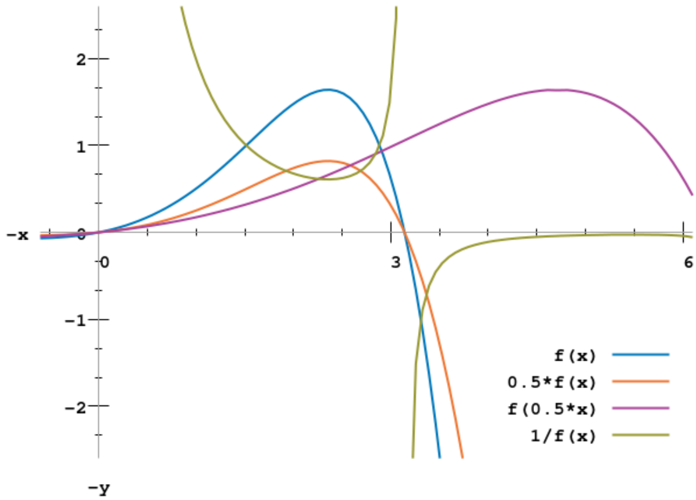
Precalculus: Functions, Geometry, Trigonometry, & Modelling

UCR Math-005-E01, Summer 2020

1. Here's a graph of a function $f(x)$.



Sketch the graphs of $\frac{1}{2}f(x)$ and $f\left(\frac{1}{2}x\right)$ and $1/f(x)$. Does f have a unique inverse function with domain $[-1, 1]$? If it does, sketch it. If not, say why not.



Notable features are that the graph of $\frac{1}{2}f(x)$ is simply less eccentric that the graph of f but still has the same x-intercepts, that the graph of $f\left(\frac{1}{2}x\right)$ is "twice as wide" as the graph of f and so the "hump" will be up at $y \approx 1.67$ and will have one x-intercept at $x = 0$ and another just past $x = 6$, and that the graph of $1/f(x)$ has asymptotes at $x = 0$ and a little past $x = 3$ and $1/f(x)$ crosses $f(x)$ everywhere $f(x) = 1$ and that $1/f(x)$ is absolutely large anytime $f(x)$ is absolutely small and vice-versa. Also f does *not* have a unique inverse with that domain since it looks like approximately $f(1.5) = 1$ and $f(2.8) = 1$, so you can uniquely define $f^{-1}(1)$.

2. (UW) Steve has a 100 m piece of wire. He wants to cut the wire into two pieces, bending one piece into a perfect circle, and bending the other piece into a perfect square. Where should Steve cut the wire so that the total amount of area enclosed by these two shapes is minimal?

The function we're trying to maximize is the area A enclosed by the two pieces of wire. On region is a square, say with side-length ℓ meters, and the other region is a circle, say with radius r meters. So the area is given by $A = \ell^2 + \pi r^2$. Now the one constraint we're given translates to the equation $4\ell + 2\pi r = 100$, which implies that $\ell = 25 - \frac{1}{2}\pi r$. Using this, we can write A purely as a function of r, like

$$\begin{aligned} A(r) &= \left(25 - \frac{1}{2}\pi r\right)^2 + \pi r^2 \\ &= \left(\frac{1}{4}\pi^2 + \pi\right) r^2 + (-25\pi) r + (100) . \end{aligned}$$

and we can see that this function is quadratic in r, and recalling the location of the vertex is given by $r = -b/2a$ where a is the coefficient of r^2 and b is the coefficient of r in A, the minimum value of A will occur at $r = \frac{50}{\pi+4}$. Backtracking to what this means in our situation, this optimum value of r is the radius of our circle. So the circumference of the circle would have to be $\frac{100\pi}{\pi+4}$. When cutting the wire Steve should use $\frac{\pi}{\pi+4}$ of the wire for the circle, and the other $1 - \frac{\pi}{\pi+4}$ for the square.

Then we could find the total enclosed area by looking at our function A again:

$$A\left(\frac{50}{\pi+4}\right) = = \left(\frac{1}{4}\pi^2+\pi\right)\left(\frac{50}{\pi+4}\right)^2 + (-25\pi)\left(\frac{50}{\pi+4}\right) + (100)$$

Which is frankly a gross number that I don't want to calculate further.

3. A fishy is swimming in the ocean with a velocity vector of $\langle\sqrt{3},2,3\rangle$. A sharky is swimming in the ocean in a direction $\langle -2,3,5\rangle$ at a speed of 38 mph.

   (a) What is the speed of the fishy?

   (b) What is the velocity vector of the sharky?

   (c) Say the fishy originated at a point $(1,1,1)$ and the sharky originated at a point $(6,7,-6)$. Write down a function that returns the distance between the fishy and the sharky after t hours.

   (a) $|\langle\sqrt{3},2,3\rangle| = \sqrt{3+4+9} = 4\,\text{mph}$

   (b) $\dfrac{38}{|\langle -2,3,5\rangle|}\langle -2,3,5\rangle = \langle -2\sqrt{38}, 3\sqrt{38}, 5\sqrt{38}\rangle$

   (c) The fishy's position is given by $(1,1,1) + t\langle\sqrt{3},2,3\rangle$, and the sharky's distance is given by $(6,7,-6) + t\langle -2\sqrt{38}, 3\sqrt{38}, 5\sqrt{38}\rangle$. So the distance between these two positions as a function of t will be

$$d(t) = \sqrt{\left(5-(\sqrt{3}+2\sqrt{38})\,t\right)^2 + \left(6+(3\sqrt{38}-2)\,t\right)^2 + \left(-7+(5\sqrt{38}-3)\,t\right)^2}.$$