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Boat A is initially at position $(2,5)$ and moves at a constant velocity of $\langle 4,1\rangle$. Boat B is at position $(8,3)$ and moves at a constant velocity of $\langle 2,10\rangle$. Do the paths of the boats ever cross? Where? Do the boats collide? What is the closest the boats ever get to each other?

Let $P_A(t)$ and $P_B(t)$ denote the position of boat A and B respectively. Explicitly these positions are given by

$$P_A(t) = (2,5) + t\langle 4,1\rangle = (2+4t,\ 5+t)$$
$$P_B(t) = (8,3) + t\langle 2,10\rangle = (8+2t,\ 3+10t).$$

To see if the paths of the boats cross, it'll be helpful to write ~~their~~ down the equations of the lines that correspond to their paths. Noting that the velocity vector of each boat gives us the slope of the line corresponding to the boat's path these equations must be

$$\text{Boat A}: \quad y-5 = \tfrac{1}{4}(x-2)$$
$$\text{Boat B}: \quad y-3 = 5(x-8).$$

After some arduous calculations, we find that these lines intersect at the point $\left(\frac{166}{19}, \frac{127}{19}\right)$. So the paths of the boats intersect at this point. But to see if the boats collide, we should calculate the time at which each boat is at this point. Boat A is at this point when the x-coordinate of its position, $2+4t$, is equal to $\frac{166}{19}$. This happens when $t = \frac{32}{19}$. Similarly we can calculate that Boat B is at this point when $t = \frac{7}{19}$, so the boats do not collide.

To calculate the closest the boats ever get to each other, it'll be helpful to write down a function ~~of time~~ that expresses the distance between the boats in terms of time. Let's name that function $d(t)$, so looking at $P_A(t)$ and $P_B(t)$ we have

$$d(t) = \sqrt{\left((2+4t)-(8+2t)\right)^2 + \left((5+t)-(3+10t)\right)^2}$$

$$= \sqrt{5(17t^2 - 12t + 8)}$$

Now finding out how close the boats get comes down to minimizing the value of $\partial(t)$, which just comes down to finding the value of t at which $17t^2-12t+8$ is minimal. Since $17t^2-12t+8$ doesn't appear to factor nicely, we need to recall that the t-value of the vertex of the parabola given by a quadratic equation at^2+bt+c is given by $-b/2a$. So the time at which the minimum occurs in our given parabola is $t = -\frac{(-12)}{2(17)} = 6/17$. Then the minumum distance between the two boats will be

$$\partial(6/17) = \sqrt{5\left(17(6/17)^2 - 12(6/17) + 8\right)}$$

$$= \sqrt{\frac{5(6^2 - 12\cdot 6 + 8\cdot 17)}{17}}$$

$$= 10\sqrt{\frac{5}{17}}$$