Just Write it Down

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The hardest part of responding to these questions is just writing it down. None of these require that you know a theorem or trick, but only that you verify an algebraic gadget has a desired structure.

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1. Let G be a group, and let A be an abelian group. Let $\varphi\colon G \to \mathrm{Aut}(A)$ be a group homomorphism. Let $A \times_\varphi G$ denote the set $A \times G$ with the binary operation

   $$(a, g)(a', g') = (a + \varphi(g)(a'), gg')\,.$$

   Prove that $A \times_\varphi G$ is a group. Why do we require that A be abelian?

2. Suppose that R and S are commutative rings and that M is a (R, S)-bimodule. This means that M is a left R-module and a right S-module and the actions are compatible, i.e. $r(ms) = (rm)s$, for all $r \in R$, $s \in S$, and $m \in M$. Let N be a left S-module. How does one define a left R-module structure on $M \otimes_S N$? What must you check to see that the action is well-defined? If we assume now in addition that N is a (S, R)-bimodule that can you say about $M \otimes_S N$? What if we don't assume that R and S are commutative?

   Suppose now that $R = \boldsymbol{k}$ is a field, so M and N are vector spaces over $\boldsymbol{k}$. If $\dim_{\boldsymbol{k}} M = m$ and $\dim_{\boldsymbol{k}} N = n$ and you are given bases for M and N, what is a natural choice of basis for $M \otimes_{\boldsymbol{k}} N$?

3. For a commutative unital ring R and left R-modules M and N, does $\mathrm{Hom}_R(M, N)$ have any sort of R-module structure? Is it necessary to assume that R is commutative? What if M is a right R-module instead?

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4. For a unital ring R and a unitary left R-module M, write out the details of the left R-module isomorphism $R \otimes_R M \cong M$.

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R-module isomorphism $M \cong \mathrm{Hom}_R(R, M)$.

6. For a left R-module M, recall the definition of the *dual* module $M^* = \mathrm{Hom}_R(M, R)$. For a ring R and left R-modules M and N, write down the details of the morphism of abelian groups
$$M^* \otimes_R N \longrightarrow \mathrm{Hom}_R(M, N)\,.$$
Prove that this homomorphism is in fact an isomorphism if R is a field and M and N are finite-dimensional vector spaces over R.

7. For integers m and n, write out the details of the $\mathbf{Z}$-bimodule isomorphism
$$\mathbf{Z}/\mathbf{Z}_m \otimes_{\mathbf{Z}} \mathbf{Z}/\mathbf{Z}_n \cong \mathbf{Z}/\mathbf{Z}_{\gcd(m,n)}\,.$$

8. Recall the definition of a *left exact functor*. Using left R modules A, B, C, and X and exact sequence $B \hookrightarrow C \twoheadrightarrow A$, write out the details that show $\mathrm{Hom}_R(X, -)$ is a left exact functor of R-modules. What can we require of X to make $\mathrm{Hom}_R(X, -)$ an exact functor?

   Now recall the definition of a *contravariant functor*. Using left R modules A, B, C, and X and exact sequence $B \hookrightarrow C \twoheadrightarrow A$, write out the details that show $\mathrm{Hom}_R(-, X)$ is a contravariant left exact functor of R-modules. What can we require of X to make $\mathrm{Hom}_R(-, X)$ an exact functor?

   For an (S, R)-bimodule X, consider the functor $(X \otimes_R -)$. What are the domain and codomain of this functor? Is it a covariant or contravariant functor? Is it left exact or right exact?

9. For a left R-module M, recall the definition of the *dual* module $M^* = \mathrm{Hom}_R(M, R)$ Write down the details of the natural homomorphism of R-modules $\theta_M\colon M \to M^{**}$. Prove that θ_M is an isomorphism if R is unital and M is free with finite basis over R.

   For a homomorphism of left R-modules $f\colon M \to N$, write down the details of the natural map $f^*\colon M^{**} \to N^{**}$ such that the following diagram commutes.

$$\begin{array}{ccc} M & \xrightarrow{\theta_M} & M^{**} \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f^*} \\ N & \xrightarrow[\theta_N]{} & N^{**} \end{array}$$

10. (EXTRA) Suppose that R and S are commutative unital rings, and we have a unital ring homomorphism $\varphi\colon R \to S$. Write down the details of the induced *restriction of scalars* functor $\Phi\colon S\text{-Mod} \to R\text{-Mod}$.

    Now S, being a left S-module itself, can be regarded as a left R-module via this functor. Write down the details of the induced *extension of scalars* functor Mod-$R \to$ Mod-S (*right* modules) given as $(- \otimes_R S)$.