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Algebra Qualifying Examination, Fall 2016, Part b

Answer any four of the following questions. All questions are worth 10 points

1. Let R be a commutative ring with identity and let a be a non–zero element in R. Suppose that P is a prime ideal properly contained in the principal ideal generated by a. Prove that $P = aP$. Suppose now that P is also principal. Prove that there exists $b \in R$ with $(1-ab)P = 0$ .What can you conclude about P if R is an integral domain and a is not a unit.

✓ 2. (a) Let R be a commutative ring with identity and regard R as a module for itself via left multiplication. Prove that this module is simple iff R is a field.
(b) Define a free module for a ring R. Suppose that R is a commutative ring with identity and satisfies the following condition: any submodule of a free module is free. Prove that R is a principal ideal domain.

✓ 3. Give examples to show that the following can happen for a ring R and modules M, N,
(i) $M \otimes_R N \ncong M \otimes_{\mathbb{Z}} N$, where $\mathbb{Z}$ is the ring of integers.
(ii) $u \in M \otimes_R N$ but $u \neq m \otimes n$ for any $m \in M$ and $n \in N$.
(iii) $u \otimes v = 0$ but $u, v \neq 0$.

✓ 4. Suppose that E is a three dimensional vector space over a field F and $f : E \to E$ is a non–zero linear transformation. Prove that there exists bases B_1 and B_2 of E such that the matrix of f is exactly one of the following.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

✓ 5. Suppose that $D = (d_1, \cdots, d_n)$ is a diagonal matrix where the d_i, $1 \leq i \leq n$ are not necessarily distinct. What are the elementary and invariant factors of D? Suppose that A is similar to D. What can you say about its elementary divisors and invariant factors?