Gan

ALGEBRA QUALIFIER 2015 - PART C

**Do 4 out of the 5 problems.**

(1) Let $K$ be a field and $f \in K[x]$. Let $n$ be the degree of $f$. Prove the theorem which states that there exists a splitting field $F$ of $f$ over $K$ with $[F:K] \leqslant n!$.

(2) Let $K$ be a subfield of $\mathbb{R}$. Let $L$ be an intermediate field of $\mathbb{C}/K$. Prove that if $L/K$ is a finite Galois extension of odd degree, then $L \subseteq \mathbb{R}$.

(3) Let $K$ be a finite field of characteristic $p$. Prove that every element of $K$ has a unique $p$-th root in $K$.

(4) Let $f(x) = x^5 - 4x + 2 \in \mathbb{Q}[x]$. Prove that $f(x) = 0$ is not solvable by radicals over $\mathbb{Q}$.

(5) Let $F/K$ be a field extension whose transcendence degree is finite. Prove that if $F$ is algebraically closed, then every $K$-monomorphism $F \to F$ is in fact an automorphism.