Greenstein

QUALIFYING EXAMINATION, ALGEBRA, PART A, 2015

September 26, 2015

Solve any four questions; indicate which ones are supposed to be graded. Each question is worth 15 points. You must show all work and justify all statements either by referring to an appropriate theorem or by providing a full solution.

1.

(a) List all isomorphism classes of abelian groups of order 120. Is there a simple group of order 120?
(b) What is the maximal number of elements of order 5 in a group of order 120?
(c) How many conjugacy classes are there in S_5?

2. Let $p > q$ be primes.

(a) Describe *all* groups of order p^2 up to an isomorphism.
(b) Show that a group of order $p^n q$, $n > 0$, is solvable.

3. The action of a group G on a set X is called *transitive* if for every $x, x' \in G$ there exists $g \in G$ such that $gx = x'$.

(a) Show that the natural action of the symmetric group S_n on the set $\{1, \ldots, n\}$ is transitive and find the stabilizer of an arbitrary element of that set.
(b) Suppose that a group G acts transitively on a set X. Prove that all subgroups $\mathrm{Stab}_G\, x$, $x \in X$ are conjugate and find $[G : \mathrm{Stab}_G\, x]$.

4. Recall that an element a of a ring is called nilpotent if $a^n = 0$ for some positive integer n. Prove the following statements for a *commutative unital ring* R:

(a) The set of all nilpotent elements in R is an ideal.
(b) R is local if and only if for all $x, y \in R$, $x + y = 1_R$ implies that x or y is a unit.
(c) If every non-unit in R is nilpotent then R is local.

5. Let $R = \mathbb{Z} \times \mathbb{Z}$ as an additive abelian group while the multiplication is defined by $(x, y) \cdot (x', y') = (xy' + yx', yy' - xx')$; then R is a commutative ring with unity $1_R = (0, 1)$. Answer the following questions (all answers must be justified).

(a) Is the ideal of R generated by $(0, 5)$ prime?
(b) Is R a domain? If so, describe its field of fractions.
(c) Choose a maximal ideal M in R and describe the localization of R at M.