Mock Algebra Qualifier, Part A

July 32nd, 2018

Do four out of the five problems.

1. Prove that a group G of order $p^n q$, where p and q are prime and $p > q$, cannot be simple.

2. Let G be a group, and let A be an abelian group. Let $\varphi\colon G \to \operatorname{Aut} A$ be a group homomorphism. Let $A \times_\varphi G$ denote the set $A \times G$ with the binary operation

$$(a, g)(a', g') = (a + \varphi(g)(a'), gg') \,.$$

(i) Prove that $A \times_\varphi G$ is a group.

(ii) Find a map $\varphi\colon \boldsymbol{Z}_2 \to \operatorname{Aut}(\boldsymbol{Z}_m)$ such that the dihedral group D_m is isomorphic to $\boldsymbol{Z}_m \times_\varphi \boldsymbol{Z}_2$. Be sure to write down the details of this isomorphism.

3. Recall that an element x of a ring is *nilpotent* if for some integer $n > 0$ we have $x^n = 0$. Also recall that we call a ring *local* if it has a unique max ideal. For a commutative unital ring R, prove that:

(i) The set of all nilpotent elements of R form an ideal.

(ii) R is local if and only if for all x and y in R, $x + y = 1$ implies that either x or y is a unit.

(iii) R is local if any nonunit of R is nilpotent.

4. Recall that a *commutator* element of a group G is an element of the form $xyx^{-1}y^{-1}$ for $x, y \in G$, and that the commutator subgroup G' is the subgroup of G generated by all the commutator elements.

(i) Prove that G' is a normal subgroup of G.

(ii) Prove that if the center of G has index n in G, then G has at most n^2 distinct commutator elements.

5. Give an example, with a proof or explanation, of each the following

(i) A Noetherian integral domain that is not a PID

(ii) An integral domain R of characteristic zero having a non-maximal prime ideal P such that the characteristic of R/P is *not* zero.

(iii) A prime ideal of $\boldsymbol{Z}[\mathfrak{i}] := \{x + y\mathfrak{i} \mid x, y \in \boldsymbol{Z}, \mathfrak{i}^2 = -1\}$.

(iv) An ideal I of $\boldsymbol{Z}[\mathfrak{i}]$ that is *not* prime such that, for the natural copy of $\boldsymbol{Z}$ in $\boldsymbol{Z}[\mathfrak{i}]$ (the "real" component), $I \cap \boldsymbol{Z}$ is prime.

Algebra B 2018. Qualifying Exam

July 32nd, 2018

Choose 4 questions out of 6. In particular, be wary of question 3.
All rings are assumed to be unital and all modules are assumed to be unitary unless specified otherwise.
All answers must be justified.

1. Let $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$, be an exact sequence of R-modules.
(a) Which part of the sequence

$$0 \longrightarrow \mathrm{Hom}_R(D, A) \longrightarrow \mathrm{Hom}_R(D, B) \longrightarrow \mathrm{Hom}_R(D, C) \longrightarrow 0\,, \qquad (*)$$

is not necessarily exact for any left R-module D? (you don't need to prove this part)
(b) Show that if the sequence (**??**) is exact for any R-module D, then the sequence $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ splits.

2. Let R be a ring and let M and N be left R-modules. Construct a natural homomorphism of abelian groups $M^* \otimes_R N \to \mathrm{Hom}_R(M, N)$. Prove that if R is a field and M and N are finite dimensional vector spaces over R, then this natural homomorphism is actually an isomorphism of R-vector spaces.

3. For a commutative unital ring R, recall that an element $e \in R$ is *idempotent* if $e^2 = e$.
(a) Prove that if $\mathrm{char}\, R \neq 2$ and if there are ideals I and J of R such that R can be written as the internal direct sum of I and J, $R = I \oplus J$, there there is some idempotent e such that $I = eR$ and $J = (1 - e)R$.
(b) Regarding the principal ideal aR as a right R-module, prove that aR is projective if and only if the annihilator $\mathrm{Ann}(a) = \{r \in R \mid ar = 0\}$ is of the form eR for some idempotent e of R.

4. For a two-sided ideal J of a ring R, let JM denote the abelian subgroup of the R-module M generated by elements of the form jm for $j \in J$ and $m \in M$. Show that JM is, in fact, an R-submodule of M. Describe the natural left R-module structure on $R/J \otimes_R M$ and show that $R/J \otimes_R M \simeq M/JM$ as left R-modules.

5. Let R be an integral domain and let A be an $n \times n$ matrix over R. Prove that if a system of linear equations $Ax = 0$ has a non-zero solution, then $\det A = 0$. Is the converse true?

6. For a commutative Noetherian ring R (so each ideal of R is finitely generated), show that each submodule of a finitely generated R-module is itself finitely generated.

Algebra Qualifying Exam

Part C

Do any 3 problems.

1. Let $f(x) = x^5 - x + 1 \in \boldsymbol{F}_5[x]$.
   (a) Prove that f has no roots in $\boldsymbol{F}_{25}$. (HINT: What polynomial identity holds for any element of $\boldsymbol{F}_{25}$?).
   (b) Determine the splitting field and full Galois correspondence for the polynomial $x^5 - x + 1$
      (i) over $\boldsymbol{F}_5$;
      (ii) over $\boldsymbol{F}_{25}$;
      (iii) over $\boldsymbol{F}_{125}$.
2. Let F be the splitting field of $f \in K[x]$ over K. Prove that if an irreducible polynomial $g \in K[x]$ has a root in F, then g splits into linear factors over F. (This result is part of a theorem characterizing normal extensions and you may not, of course, quote this theorem or its corollaries).
3. Disprove (by example) or prove the following: If $K \to F$ is an extension (not necessarily Galois) with $[F : K] = 6$ and $\mathrm{Aut}_K(F)$ isomorphic to the Symmetric group S_3, then F is the splitting field of an irreducible cubic in $K[x]$.
4. If $\boldsymbol{Z}_p \to F$ is a field extension of degree n then the map $x \mapsto x^p$ is a $\boldsymbol{Z}_p$-automorphism of F of order exactly n whose fixed field is $\boldsymbol{Z}_p$.