Mock Algebra Qualifier, Part A
September n, 2018

Do four out of the five problems.

1. (i) Prove that if $\mathrm{Aut}(G)$ is cyclic, then G is abelian.
(ii) Prove that if $|G|$ is finite and $\mathrm{Aut}(G)$ is cyclic, then $|\mathrm{Aut}(G)|$ must be even.
(iii) Prove that there is no group with infinite cyclic automorphism group.

2. Given a finite p-group G, prove that G has a normal subgroup of every order dividing $|G|$.

3. Let R be a unital integral domain. For a nonzero element of $s \in R$, let $S = \{1, s, s^2, \dots\}$. Prove that $S^{-1}R \simeq R[x]/(xs - 1)$.

4. For a set X let $\mathcal{P}(X)$ denote the set of a subsets of X. For $A, B \in \mathcal{P}(X)$ define the operations $AB := A \cap B$ and $A + B := (A \cup B) \setminus (A \cap B)$ (the *symmetric difference* of A and B).
(i) Prove that $\mathcal{P}(X)$ is a commutative unital ring under these operations.
(ii) What is the characteristic of this ring? Prove that every ring R with the property that $AA = A$ for all $A \in R$ must have this characteristic.
(iii) Prove that every finitely generated ideal of $\mathcal{P}(X)$ is principal.

5. (i) Prove that a finite integral domain is a field. Is it true that a finite integral ring (non-commutative) is a division ring? (FUN FACT: every finite division ring is a field. This is part of Wedderburn's little theorem.)
(ii) Does there exist a field such that its additive group structure and its multiplicative group of units are isomorphic?

Algebra B 2018. Qualifying Exam

Choose 4 questions out of 6.
All rings are assumed to be unital and all modules are assumed to be unitary unless specified otherwise.

1. Recall that a functor is exact if it takes short exact sequences to short exact sequences.

(i) Prove that if F is a finite dimensional free R-module, then $-\otimes_R F$ is an exact functor.

(ii) Prove that if P is a finitely generated projective R-module, then $-\otimes_R P$ is an exact functor.

(iii) (Challenge) Prove that if R is a ring $\mathcal{P}(X)$ like in Question 4, Part A of this exam, then $-\otimes_R M$ is exact for *any* R-module M.

2. Let V and Ω be finite dimensional vector spaces over a field $\boldsymbol{k}$ and let W, W' be subspaces of V.

(i) Prove that $\dim_{\boldsymbol{k}} V = \dim_{\boldsymbol{k}} W + \dim_{\boldsymbol{k}}(V/W)$.

(ii) For a homomorphism $\varphi\colon V \to \Omega$, prove that $\dim_{\boldsymbol{k}} V = \dim_{\boldsymbol{k}}(\operatorname{Ker}\varphi) + \dim_{\boldsymbol{k}}(\operatorname{Im}\varphi)$.

(iii) Prove that $\dim_{\boldsymbol{k}} W + \dim_{\boldsymbol{k}} W' = \dim_{\boldsymbol{k}}(W \cap W') + \dim_{\boldsymbol{k}}(W + W')$.

3. Let B be an abelian group. Prove that for any subgroup A of B, a homomorphism A to $\mathbb{Q}/\mathbb{Z}$ must extend to a homomorphism B to $\mathbb{Q}/\mathbb{Z}$.

4. Let $R = \mathbb{C}[x]$.

(i) Let M be a torsion-free module for R with two generators. Prove that M is free of rank at most two.

(ii) Prove that if M is a cyclic R-module and $M \neq R$, then M is torsion. Under what condition on the torsion ideal will M be simple?

5. For a finitely generated $\mathbb{R}[x]$-module M, recall that for $f \in \mathbb{R}[x]$, we have submodules $fM = \{fm \mid m \in M\}$ and $M[f] = \{m \in M | fm = 0\}$.

(i) For an irreducible polynomial $\mathfrak{f} \in \mathbb{R}[x]$, prove that $M[\mathfrak{f}]$ admits the structure of a vector space over $\mathbb{R}[x]/(\mathfrak{f})$.

(ii) Let $f = x^2 + 1$ and $g = x + 1$. In terms of their invariant factors, describe all the isomorphism classes of $\mathbb{R}[x]$-modules M such that $\dim_{\mathbb{R}[x]} M = 5$ and that $f^r M = 0$ and $g^s M = 0$ for some $r, s > 0$.

6. For a field $\boldsymbol{k}$, let $f \in \boldsymbol{k}[x]$ be a monic polynomial. Prove that f is the minimal polynomial of it's companion matrix. Write down the companion matrix of the polynomial $x^3 - x^2 + 2x - 1$.

Fields Qualifier 2018

Do any 3 problems.

1. Let $f = x^3 - x + 1 \in \boldsymbol{F}_3[x]$. Show that f is irreducible over $\boldsymbol{F}_3$. Let K be the splitting field of f over $\boldsymbol{F}_3$. Compute the degree $[L : \boldsymbol{F}_3]$ and the number of elements of L.

2. Let $F = \mathbb{C}\left(t^4\right) \subset K = \mathbb{C}(t)$, where t is a formal variable. Compute the Galois group $\mathrm{Aut}_F(K)$, and determine it's subgroups and corresponding intermediate fields.

3. Prove that in a finite field of characteristic p every element has a unique p^{th} root. Provide an example of an infinite field of characteristic p where this is not true.

4. Let F_{12} be a cyclotomic extension of $\mathbb{Q}$ of order 12. Determine $\mathrm{Aut}_{\mathbb{Q}}(F_{12})$ and all intermediate fields.