MOCK QUALIFYING EXAMINATION, ALGEBRA, PART A, 2019

September n, 2019

Solve any four questions; indicate which ones are supposed to be graded. Each question is worth 15 points. You must show all work and justify all statements either by referring to an appropriate theorem or by providing a full solution.

1.

(a) List all isomorphism classes of abelian groups of order 120. Is there a simple group of order 120?

(b) What is the maximal number of elements of order 5 in a group of order 120?

(c) How many conjugacy classes are there in S_5?

2. Let $p > q$ be primes.

(a) Describe *all* groups of order p^2 up to an isomorphism.

(b) Show that a group of order $p^n q, n > 0$, is solvable.

3. The action of a group G on a set X is called *transitive* if for every $x, x' \in X$ there exists a $g \in G$ such that $gx = x'$.

(a) Show that the natural action of the symmetric group S_n on the set $\{1, \ldots, n\}$ is transitive and find the stabilizer of an arbitrary element in that set.

(b) Suppose that a group G acts transitively on a set X. Prove that all the subgroups $\mathrm{Stab}_G x$, $x \in X$ are conjugate and find $[G : \mathrm{Stab}_G x]$.

4. Recall that an element a of a ring is called nilpotent if $a^n = 0$ for some positive integer n. Prove the following statements for an *commutative unital ring* R.

(a) The set of all nilpotent elements in R is an ideal.

(b) R is local if and only if for all $x, y \in R$, $x + y = 1_R$ implies that x or y is a unit.

(c) If every non-unit in R is nilpotent then R is local.

5. Let $R = \mathbb{Z} \times \mathbb{Z}$ as an additive abelian group while the multiplication is defined by $(x, y) \cdot (x', y') = (xy' + yx', yy' - xx')$; then R is a commutative ring with unity $1_R = (0, 1)$. Answer the following questions (all answers must be justified).

(a) Is the ideal of R generated by $(0, 5)$ prime?

(b) Is R a domain? If so, describe it's field of fractions.

(c) Choose a maximal ideal M in R and describe the localization of R at M.

Mock Algebra Qualifying Examination, Fall 2019, Part b

Answer any four of the following questions. All questions are worth 10 points.

1. Let R be a commutative ring with identity and let a be a non-zero element in R. Suppose that P is a prime ideal properly contained in the principal ideal generated by a. Prove that $P = aP$. Suppose now that P is also principal. Prove that there exists $b \in R$ with $(1 - ab)P = 0$. What can you conclude about P if R is an integral domain and a is not a unit?

2. (a) Let R be a commutative ring with identity and regard R as a module for itself via left multiplication. Prove that this module is simple iff R is a field. (b) Define a free module for a ring R. Suppose that R is a commutative ring with identity and satisfies the following condition: any submodule of a free module is free. Prove that R is a principal ideal domain.

3. Give examples to show that the following can happen for a ring R and modules M,N,
(i) $M \otimes_R N \not\cong M \otimes_{\mathbf{Z}} N$, where $\mathbf{Z}$ is the ring of integers.
(ii) $u \in M \otimes_R N$ but $u \neq m \otimes n$ for any $m \in M$ and $n \in N$.
(iii) $u \otimes v = 0$ but $u, v \neq 0$.

4. Suppose that E is a three dimensional vector space over a field F and $f : E \to E$ is a non-zero linear transformation. Prove that there exists bases B_1 and B_2 of E such that the matrix of f is exactly one of the following.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

5. Suppose that $D = (d_1, \ldots, d_n)$ is a diagonal matrix where the d_i, $1 \leq i \leq n$ are not necessarily distinct. What are the elementary and invariant factors of D? Suppose that A is similar to D. What can you say about its elementary divisors and invariant factors?

Mock Algebra Qualifier 2019 - Part C

Do 4 out of the 5 problems.

(1) Let K be a field and $f \in K[x]$. Let n be the degree of f. Prove the theorem which states that there exists a splitting field F of f over K with $[F : K] \leqslant n!$.

(2) Let K be a subfield of $\mathbb{R}$. Let L be an intermediate field of $\mathbb{C}/K$. Prove that if L/K is a finite Galois extension of odd degree, then $L \subseteq \mathbb{R}$.

(3) Let K be a finite field of characteristic p. Prove that every element of K has a unique p-th root in K.

(4) Let $f(X) = x^5 - 4x + 2 \in \mathbb{Q}[x]$ Prove that $f(x) = 0$ is not solvable by radicals over $\mathbb{Q}$.

(5) Let F/K be a field extension whose transcendence degree is finite. Prove that if F is algebraically closed, then every K-monomorphism $F \to F$ is in fact an automorphism.