MOCK QUALIFYING EXAMINATION, ALGEBRA, PART A, 2019

September n^3, 2019

Solve any four questions; indicate which ones are supposed to be graded. You must show all work and justify all statements either by referring to an appropriate theorem or by providing a full solution.

1. Let $G = \mathbb{Q}/\mathbb{Z}$, where $\mathbb{Q}$ and $\mathbb{Z}$ are considered as additive groups. Prove that for any positive integer n, G has a unique subgroup $G(n)$ of order n, and that $G(n)$ is cyclic.

2. For groups $N_1 \trianglelefteq G_1$ and $N_2 \trianglelefteq G_2$, provide a counterexample to each of the following statements.

(a) $G_1 \cong G_2$ and $N_1 \cong N_2$ implies that $G_1/N_1 \cong G_2/N_2$.

(b) $G_1 \cong G_2$ and $G_1/N_1 \cong G_2/N_2$ implies that $N_1 \cong N_2$.

(c) $N_1 \cong N_2$ and $G_1/N_1 \cong G_2/N_2$ implies that $G_1 \cong G_2$.

3. Let R be a unital integral domain. For a nonzero element of $s \in R$, let $S = \{1, s, s^2, \dots\}$. Prove that $S^{-1}R \cong R[x]/(xs - 1)$.

4. Given a finite p-group G, prove that G has a normal subgroup of every order dividing $|G|$.

5.

(a) Define the characteristic of a ring.

(b) Assume that R is a commutative unitary ring having only one maximal ideal $\mathfrak{m}$. Show that the characteristic of R is either zero or a power of a prime.

(c) For R as described in (b) show that if $R/\mathfrak{m}$ has characteristic zero, then R contains a field.

(d) Give an example of a ring R as in (b) of characteristic zero having a non-maximal prime ideal P such that the characteristic of R/P is *not* zero.

Mock Algebra Qualifying Examination, Fall 2019, Part b

Attempt any four, all questions are worth 10 points.

1. (a) Let R be a ring with identity and M a left module for R. Recall that M is indecomposable if M cannot be written as a direct sum of two non-zero submodules. Prove that if $f\colon M \to M$ is a homomorphism of modules then $f^2 = f$ implies that either $f = 0$ or $f = id$.

(b) Suppose now that M is decomposable. Prove that there exists $f\colon M \to M$ a homomorphism of modules such that $f^2 = f$ and f different from zero and the identity.

2. Suppose R is a ring with identity and $e \in R$ such that $e^2 = e$.

(a) Prove that $(1 - e)$ has the same property.

(b) Prove that $Re \cap R(1 - e) = \{0\}$, and hence $R = Re \oplus R(1 - e)$.

(c) Regarding the principal ideal Ra as a left R-module, prove that Ra is projective if and only if the annihilator $\operatorname{Ann}(a) = \{r \in R \mid ra = 0\}$ is of the form Re for some idempotent e of R.

3. Let R be a ring with identity. Regard R as a right R-module in the usual way and let M be a right R module. Prove that $\operatorname{Hom}_R(R, M) \cong M$ as abelian groups.

4. Consider the ring $R = \mathbf{C}[x]$ of polynomials in an indeterminate x with coefficients in $\mathbf{C}$.

(a) Let M be a torsion free module for R with two generators. Prove that M is free of rank at most two.

(b) Prove that if M is a cyclic R-module and $M \neq R$ then M is torsion. Under what condition on the torsion ideal will M be simple?

5. (a) Prove that if A and B are invertible $n \times n$ matrices with entries in an integral domain R, then $A + rB$ is invertible in the quotient field K of R for all but finitely many r.

(b) Prove that the minimal polynomial of a linear transformation of an n-dimensional vector space has degree at most n.

6. Suppose that φ and ψ are commuting linear transformations of an n-dimensional vector space E. Prove that if E_1 is a ~~φ-invariant subspace of E~~ eigenspace of φ then E_1 is ~~also~~ ψ-invariant. Use this to prove that if φ and ψ both have linear elementary divisors then there exists a basis of E with respect to which the matrix φ and the matrix ψ are both diagonal.

Mock Algebra Qualifier 2019 - Part C

Do 4 out of the 5 problems.

(1) Let F be a splitting field over $\mathbf{Q}$ of the polynomial $x^4 - 5$. Find all the intermediate fields of F over $\mathbf{Q}$, and indicate which ones are Galois over $\mathbf{Q}$.

(2) Prove that $\mathbf{Q}(\sqrt{2} + \sqrt{3}) = \mathbf{Q}(\sqrt{2}, \sqrt{3})$

(3) Let F be the splitting field of $f \in K[x]$ over K. Prove that if an irreducible polynomial $g \in K[x]$ has a root in F, then g splits into linear factors over F. (This result is part of a theorem characterizing normal extensions and you may not, of course, quote this theorem or its corollaries).

(4) Let p be a prime and n be any natural number.
  (a) Prove that there exists an irreducible polynomial f of degree n in $\mathbf{Z}_p[x]$.
  (b) Let $f \in \mathbf{Z}_p[x]$ be an irreducible polynomial of degree n. Determine with proof the degree of the splitting field of f over $\mathbf{Z}_p$.
  (c) Exhibit with proof irreducible polynomials of degree 2, 3, and 4 over $\mathbf{Z}_2$.

(5) Let $\boldsymbol{F}_7$ be a cyclotomic extension of $\mathbf{Q}$ of order seven. If ζ is a primitive seventh root of unity, what is the irreducible polynomial over $\mathbf{Q}$ of $\zeta + \zeta^{-1}$? You must justify your answer.