MOCK QUALIFYING EXAMINATION, ALGEBRA, PART A, 2019

September n^4, 2019

Solve any four questions; indicate which ones are supposed to be graded. You must show all work and justify all statements either by referring to an appropriate theorem or by providing a full solution.

1. For a group G, let G' denote its commutator subgroup.
(a) Prove that G' is normal in G.
(b) Show that for any abelian group A, a homomorphism $G \to A$ must factor through the quotient G/G'.
(c) Let $G^{(1)} = G'$, $G^{(2)} = (G')'$, and in general $G^{(n)} = (G^{(n-1)})'$. Give an example of a group G such that $G^{(n)} \neq \langle e \rangle$ for any $n \in \mathbf{N}$.

2. Classify all groups of order 169.

3. An integral domain R is integrally closed if for any monic polynomial f over R, every root of f in $\text{Frac}(R)$ is actually in R.
(a) Prove that a unique factorization domain is integrally closed.
(b) Give an example of a ring that is *not* integrally closed.

4.
(a) Prove that a finite integral domain is a field. Is it true that a finite integral ring (non-commutative) is a division ring?
(b) Does there exist a field such that its additive group structure and its multiplicative group of units are isomorphic?
(c) (Challenge) Prove that every finite division ring is a field.

5. For a set X let $\mathcal{P}(X)$ denote the set of a subsets of X. For $A, B \in \mathcal{P}(X)$ define the operations $AB := A \cap B$ and $A + B := (A \cup B) \setminus (A \cap B)$ (the *symmetric difference* of A and B).
(a) Prove that $\mathcal{P}(X)$ is a commutative unital ring under these operations.
(b) What is the characteristic of this ring? Prove that every ring R with the property that $AA = A$ for all $A \in R$ must have this characteristic.
(c) Prove that every finitely generated ideal of $\mathcal{P}(X)$ is principal.

Mock Algebra Qualifying Examination, Fall 2019, Part b

Attempt any four, all questions are worth 10 points.

1. (a) Prove that every quotient of a divisible group is divisible.

(b) Let B be an abelian group. Prove that for any subgroup A of B, a homomorphism A to $\mathbf{Q}/\mathbf{Z}$ must extend to a homomorphism B to $\mathbf{Q}/\mathbf{Z}$.

2. For a ring R, consider the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & B & \overset{i_1}{\rightarrowtail} & C & \overset{\pi_1}{\twoheadrightarrow} & A & \longrightarrow & 0 \\ & & & & \downarrow{\scriptstyle h} & & & & \\ 0 & \longrightarrow & Y & \underset{i_2}{\rightarrowtail} & Z & \underset{\pi_2}{\twoheadrightarrow} & X & \longrightarrow & 0 \end{array}$$

in the category of R-modules such that the top and bottom rows are exact.

(a) Suppose that there is a map $g \in \mathrm{Hom}_R(B, Y)$ such that $hi_1 = i_2 g$. Prove that there exists a map $f \in \mathrm{Hom}_R(A, X)$ such that $f\pi_1 = \pi_2 h$.

(b) Now suppose that there exists some map $f \in \mathrm{Hom}_R(A, X)$ such that $f\pi_1 = \pi_2 h$. Does there necessarily exist a map $g \in \mathrm{Hom}_R(B, Y)$ such that $hi_1 = i_2 g$?

3. Let V be a finite dimensional vector space over $\mathbf{C}$, and take φ in $\mathrm{End}_{\mathbf{C}}(V)$.

(a) Prove that φ defines a left $\mathbf{C}[x]$-module structure on V where, for $f \in \mathbf{C}[x]$ and $\boldsymbol{v} \in V$, $f(\varphi) \in \mathrm{End}_{\mathbf{C}}(V)$ and $f.\boldsymbol{v} := \big(f(\varphi)\big)(\boldsymbol{v})$.

(b) We say a subspace $W \subset V$ is *φ-invariant* if $\varphi(W) \subset W$. Prove that W is φ-invariant if and only if W is a $\mathbf{C}[x]$-submodule of V under the action inducted by φ. Furthermore prove that $V_\varphi(\boldsymbol{v})$, the smallest φ-invariant subspace of V containing $\boldsymbol{v}$, is the cyclic submodule $\mathbf{C}[x]\boldsymbol{v}$.

4. Consider the matrices

$$M = \begin{pmatrix} 0 & 0 & 0 & 5 \\ 0 & 0 & 2 & 0 \\ 0 & 2 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \qquad N = \begin{pmatrix} 0 & 0 & 0 & 2 \\ 0 & 0 & 5 & 0 \\ 0 & 2 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

(a) What are the invariant factor and elementary divisor decompositions of the $\mathbf{Q}[x]$-module corresponding to M? What are these decompositions if you consider the corresponding $\mathbf{C}[x]$-module instead? What about the decomposition as a $\boldsymbol{F}_5[x]$-module where $\boldsymbol{F}_5$ is the field with five elements?

(b) What is the Jordan canonical form of M considered as a matrix over $\mathbf{C}$? What is the Jordan canonical form over $\overline{\boldsymbol{F}_5}$, the algebraic closure of $\boldsymbol{F}_5$?

(c) Determine, with proof, whether or not the matrices M and N are equivalent over $\mathbf{C}$. Are M and N similar over $\mathbf{C}$? Are M and N similar over $\boldsymbol{F}_5$?

6. Recall that a functor is exact if it takes short exact sequences to short exact sequences.

(a) Prove that if F is a finite dimensional free R-module, then $- \otimes_R F$ is an exact functor.

(b) Prove that if P is a finitely generated projective R-module, then $- \otimes_R P$ is an exact functor.

(c) (Challenge) Prove that if R is a ring $\mathcal{P}(X)$ like in Question 5, Part A of this exam, then the functor $- \otimes_R M$ is exact for *any* R-module M.

MOCK ALGEBRA QUALIFIER 2019 - PART C

Do 4 out of the 5 problems.

(1) Let F/k be a normal extension of fields and let K_0 be the maximal separable subextension of k. Show that K_0/k is normal.

(2) Let F be a field and $p(x) \in F[x]$ an irreducible polynomial.
  (a) Prove that there exists a field extension K of F in which $p(x)$ has a root.
  (b) Determine the dimension of K as a vector space over F and exhibit a vector space basis for K.
  (c) If $\theta \in K$ denotes a root of $p(x)$, express θ^{-1} in terms of the basis found in part (b).
  (d) Suppose $p(x) = x^3 + 9x + 6$. Show $p(x)$ is irreducible over $\mathbf{Q}$. If θ is a root of $p(x)$, compute the inverse of $(1 + \theta) \in \mathbf{Q}(\theta)$.

(3) Let $f = x^5 - 45x^3 + 35x^2 + 15$ and $g = x^{11} - 11$, both considered as polynomials in $\mathbf{Q}[x]$. Suppose $\alpha \in \mathbf{C}$ is a root of f. Prove or disprove: $\mathbf{Q}(\alpha)$ contains a root of g.

(4) Given a tower of fields $F \to E \to K$, prove or disprove by providing a counterexample:
  (a) If K is normal over F, then K is normal over E.
  (b) If K is normal over E and E is normal over F, then K is normal over F.
  (c) If K is separable over F, then K is separable over E and E is separable over F.

(5) Let p be a prime number and $K = \boldsymbol{F}_{p^6}$ be a field with p^6 elements.
  (a) Given an element of K, what are the possible degrees of it's minimal polynomial over $\boldsymbol{F}_p$?
  (b) For each possible degree, how many elements in K have a minimal polynomial with that degree?