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SINGULARITY FORMATION OF EMBEDDED CURVES EVOLVING ON SURFACES BY CURVATURE FLOW

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Abstract: In this paper, we extend Grayson's Theorem [16] on curvature flow of embedded curves in a compact Riemannian surface. The main result is a direct and shorter proof of a theorem of X. Zhu [25] that, if a singularity develops in finite time, then the curve converges to a round point in a C^∞ sense. The proof will extend Hamilton's isoperimetric estimates technique for curvature flow of embedded curves in the plane [18].

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1. Introduction

Let γ be a closed embedded curve evolving under the curvature flow in a compact surface M. If a singularity develops in finite time, then the curve shrinks to a point [16]. So when t is close enough to the blow-up time ω, we may assume that the curve is contained in a small neighborhood of the collapsing point on the surface. Using a local conformal diffeomorphism $\phi : U(\subseteq M) \to U' \subseteq \mathbb{R}^2$

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between compact neighborhoods, we get a corresponding flow in the plane which satisfies the following equation:

$$\frac{\partial \gamma'}{\partial t} = \left(\frac{k'}{J^2} - \frac{\nabla_N J}{J^2} \right) N', \tag{1}$$

where $\gamma'(p,t) = \phi(\gamma(p,t))$, k' is the curvature of γ' in U', N' is the unit normal vector, and the conformal factor J is smooth, bounded and bounded away from 0.

For a smooth embedded closed curve γ in $\mathbb{R}^2$, consider any curve Γ which divides the region enclosed by γ into two pieces with areas A_1 and A_2, where $A_1 + A_2 = A$ is the area enclosed by γ. Let L be the length of Γ. Define [18],

$$G(\gamma, \Gamma) = L^2 \left(\frac{1}{A_1} + \frac{1}{A_2} \right), \quad \text{and} \quad \overline{G}(\gamma) = \inf_{\Gamma} G(\gamma, \Gamma).$$

We use the above isoperimetric estimates to prove the following theorem.

Main Theorem. *Let γ be a closed embedded curve evolving by curvature flow on a smooth compact Riemannian surface. If a singularity develops in finite time, then the curve converges to a round point in the C^∞ sense.*

2. Evolving Closed Curves in a Surface

Grayson [16] and Gage [14], generalized the study of curvature flow of closed curves in the plane to that in surfaces. The *curvature flow* is a gradient flow for the length functional on the space of immersed curves in the surface M^2 with Riemannian metric g.

Let (M, g) be a smooth compact oriented 2-dimensional Riemannian manifold with bounded scalar curvature. Let $\gamma_0 : S^1 \to M$ be a smooth embedded curve in M and let $\gamma : S^1 \times [0, \omega) \to M$ be a one-parameter smooth family of embedded curves satisfying $\gamma(\cdot, 0) = \gamma_0$. If γ evolves by curvature flow, then

$$\frac{\partial \gamma}{\partial t}(p,t) = k(p,t)N(p,t), \quad (p,t) \in S^1 \times [0, \omega), \tag{2}$$

where k is the geodesic curvature of γ and N is its unit normal.

Arclength is given by

$$s(p,t) = \int_0^p \left| \frac{\partial \gamma}{\partial q}(q,t) \right| dq.$$

Differentiating,

$$\frac{\partial s}{\partial p}(p,t) = \left|\frac{\partial \gamma}{\partial p}(p,t)\right| = v(p,t)$$

$$\Rightarrow \frac{\partial}{\partial s} = \frac{1}{v}\frac{\partial}{\partial p}, \quad \text{and} \quad ds = vdp.$$

From the Frenet formulas, we have

$$\nabla_s T = kN \quad \text{and} \quad \nabla_s N = -kT.$$

Now we recall some standard results for the evolution, see [16].

Lemma 2.1. *For the curvature flow:*

1. *The speed v evolves according to $\frac{\partial v}{\partial t} = -k^2 v$.*
2. $[\frac{\partial}{\partial t}, \frac{\partial}{\partial s}] = k^2 \frac{\partial}{\partial s}$.
3. $\nabla_t T = \frac{\partial k}{\partial s} N$ *and* $\nabla_t N = -\frac{\partial k}{\partial s} T$.
4. *The arclength L of the curve evolves according to $\frac{dL}{dt} = -\int_{\gamma_t} k^2 ds$.*
5. $\nabla_t \nabla_s = \nabla_s \nabla_t + k^2 \nabla_s - kR(T,N)$.
6. *The curvature k of the curve evolves according to $\frac{\partial k}{\partial t} = \frac{\partial^2 k}{\partial s^2} + k^3 + K\,k$, where $K = \langle R(N,T)T, N\rangle$ is the Gaussian curvature of M restricted to $\gamma(\cdot,t)$.*

Theorem 2.1. (see [16]) *A closed embedded curve moving on a smooth compact Riemannian surface by curvature flow must either collapse to a point in finite time or else converge to a simple closed geodesic as $t \to \infty$.*

Grayson's proof was rather delicate, requiring separate analyses of what may happen under various geometric configurations, and special arguments for each cases. First he showed that the solution remains smooth and embedded as long as its curvature remains bounded. He then proved that if a singularity develops in finite time, then the curvature remains bounded until the entire curve shrinks to a point. Finally, he proved that if the length of the curve does not converge to zero, then its curvature must converge to zero in the C^∞ norm and that the curve approaches a geodesic in the C^∞ sense.

In this paper, the proof has been simplified using Hamilton's isoperimetric estimates technique from [18] to rule out certain kinds of singularity and we extend Grayson's Theorem [16] by showing that if the curve shrinks to a point, then it shrinks to a round point in a C^∞ sense. Since the curve does shrink to a point, we can transform the curvature flow in surfaces to a corresponding flow in the plane (more general than the curvature flow in the plane per se). In a series of papers, Angenent [4], [6], [5] developed a more general theory of parabolic equations for curves on surfaces. We now summarize some of the important results of Angenent that we will need.

2.1. Parabolic Equations for Curves on Surfaces

Consider a closed curve evolving by an arbitrary uniformly parabolic equation,

$$\frac{\partial \gamma}{\partial t} = V(T,k)N, \tag{3}$$

on a smooth oriented 2-dimensional Riemannian manifold M, and denote its unit tangent bundle by $S^1(M) = \{\xi \in T(M) : g(\xi,\xi) = 1\}$. Then the normal velocity is

$$v^{\perp}(p,t) = V(T,k)(p,t) \equiv V(T_{\gamma(p,t)}, k_{\gamma(p,t)}),$$

for some function $V : S^1(M) \times \mathbb{R} \to \mathbb{R}$ which satisfies:

$$\begin{aligned}
&(V_1) \quad V(T,k) \text{ is } C^{2,1},\\
&(V_2) \quad \lambda^{-1} \le \frac{\partial V}{\partial k} \le \lambda,\\
&(V_3) \quad |V(T,0)| \le \mu \quad \text{for all } T \in S^1(M),\\
&(V_4) \quad |\nabla^h V| + |k\nabla^v V| \le \nu(1+k^2),\\
&(V_5) \quad V(-T,-k) = -V(T,k),
\end{aligned}$$

for positive constants λ, μ, and ν.

The tangent bundle to $S^1(M)$ splits into the Whitney sum of the bundle of horizontal vectors and bundle of vertical vectors. $\nabla^v V$ and $\nabla^h V$ denote the vertical and horizontal components of $\nabla(V)$ (holding the second argument of V fixed).

These assumptions on V are necessary to make the set of allowable initial curves as large as possible, and necessary for the short-time existence of the solutions. The way in which maximal classical solutions can become singular (limit curves) is based on these assumptions on V and the initial curves. Our application of this theory will be for the flow given by $V(T,k) = \left(\frac{k}{J^2} - \frac{\nabla_N J}{J^2}\right)$, where $J(x,y)$ is a smooth bounded function that is also bounded away from 0. We will see that the curvature flow in a surface corresponds to the flow with this normal velocity in a plane.

Lemma 2.2. *For the flow (3):*

1. *The speed v evolves according to $\frac{\partial v}{\partial t} = -kVv$.*
2. *$[\frac{\partial}{\partial t}, \frac{\partial}{\partial s}] = kV\frac{\partial}{\partial s}$.*

3. $\nabla_t T = \frac{\partial V}{\partial s} N$, *and* $\nabla_t N = -\frac{\partial V}{\partial s} T$.

4. The arclength L *of the curve evolves according to* $\frac{dL}{dt} = -\int_{\gamma_t} kV\,ds$.

5. $\nabla_t \nabla_s = \nabla_s \nabla_t + kV\nabla_s - VR(T,N)$.

6. The curvature k *of the curve evolves according to* $\frac{\partial k}{\partial t} = \frac{\partial^2 V}{\partial s^2} + k^2 V + KV$, *where* $K = \langle R(N,T)T, N\rangle$ *is the Gaussian curvature of* M *restricted to* $\gamma(\cdot,t)$.

7. The enclosed area A *of the curve evolves according to* $\frac{dA}{dt} = -\int_{\gamma_t} V\,ds$.

We now state the main result from [4] and [6].

Theorem 2.2. *Let* V *satisfy* $(V_1)-(V_5)$, *and let* $\gamma : S^1 \times [0,\omega) \to M$ *be a maximal classical solution of (3) which becomes singular in finite time. Then the limit curve* γ^* *of the* $\gamma(\cdot,t)$ *either has fewer self-intersections than any of the* $\gamma(\cdot,t)$*'s, or else the total absolute curvature of the limit curve drops by at least* π.

Oaks [23] improved Theorem 2.2 by showing that the latter case never occurs. So if the initial curve is embedded, and the singularity develops in finite time, then the curve shrinks to a point. So when t is close enough to the blow-up time ω, we may assume that the curve is contained in a small neighborhood of the collapsing point on the surface.

Now from the following theorem, it is enough to work locally in $\mathbb{R}^2$.

Theorem 2.3. (see [23]) *Let* $\phi : U(\subseteq M) \to U' \subseteq \mathbb{R}^2$ *be a conformal diffeomorphism between compact neighborhoods. If* $V : S^1(M) \times \mathbb{R} \to \mathbb{R}$ *satisfies* $(V_1)-(V_5)$, *then there is a function* $V' : S^1(U') \times \mathbb{R} \to \mathbb{R}$ *which satisfies* $(V_1)-(V_5)$ *such that whenever* $\gamma(p,t)$ *is a curve in* U *evolving by (3),* $\gamma'(p,t) = \phi(\gamma(p,t))$ *satisfies* $\frac{\partial \gamma'}{\partial t} = V'(T',k')N'$, *where* T' *and* N' *are the unit tangent and normal vectors, and* k' *is the curvature of* γ' *in* U'.

Moreover, $V(T,k) = J(p)V'(T',k')$ *and* $ds = J(p)ds'$, *where* $J(p) > 0$ *is smooth, bounded, and bounded away from* 0.

The metric in U can be written as

$$g = J^2(x,y)(dx^2 + dy^2),$$

where the coordinates in U are obtained by ϕ^{-1}. Because U' is compact, $J(x,y)$ is both bounded and bounded away from 0.

Let $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ be the coordinate vector fields on U, and let $X = \frac{1}{J}\frac{\partial}{\partial x}$, and $Y = \frac{1}{J}\frac{\partial}{\partial y}$. Then X and Y are unit vectors. Since ϕ is conformal, $\phi_*(N) = \frac{1}{J}N'$.

So γ' evolves by the equation:

$$\frac{\partial \gamma'}{\partial t} = (\frac{1}{J}V)N'.$$

Therefore, $V' = \frac{1}{J}V$.

We next show that $k' = kJ + \nabla_N J$. First, we need the following lemma.

Lemma 2.3.

$$\nabla_X X = -\frac{\nabla_Y J}{J}Y, \qquad \nabla_X Y = \frac{\nabla_Y J}{J}X$$
$$\nabla_Y X = \frac{\nabla_X J}{J}Y, \qquad \nabla_Y Y = -\frac{\nabla_X J}{J}X.$$

Proof. Since $0 = [\frac{\partial}{\partial x}, \frac{\partial}{\partial y}] = [JX, JY]$, we have $\nabla_{JX} JY = \nabla_{JY} JX$. Therefore, $J\nabla_X Y + (\nabla_X J)Y = J\nabla_Y X + (\nabla_Y J)X$. Since $\nabla_X Y \perp Y$ and $\nabla_Y X \perp X$, we get $\nabla_X Y = \frac{\nabla_Y J}{J}X$ and $\nabla_Y X = \frac{\nabla_X J}{J}Y$. The other two formulas follow from differentiating $\langle X, Y\rangle = 0$ with respect to X and Y. $\square$

Let θ be the angle T makes with X in U. Then

$$T = \cos\theta X + \sin\theta Y, \quad N = -\sin\theta X + \cos\theta Y.$$

Thus,

$$\begin{aligned}\nabla_T X &= -\cos\theta \frac{\nabla_Y J}{J}Y + \sin\theta \frac{\nabla_X J}{J}Y \\ &= \left(-\frac{\nabla_Y J}{J}\cos\theta + \frac{\nabla_X J}{J}\sin\theta\right)Y,\end{aligned}$$

and

$$\nabla_T Y = \left(\frac{\nabla_Y J}{J}\cos\theta - \frac{\nabla_X J}{J}\sin\theta\right)X.$$

We have $\nabla_T \theta = \frac{1}{J}k'$. Then

$$\begin{aligned}kN = \gamma'' &= \nabla_T T = \nabla_T(\cos\theta\, X + \sin\theta\, Y) \\ &= -\sin\theta(\frac{k'}{J})X + \cos\theta\left(-\frac{\nabla_Y J}{J}\cos\theta + \frac{\nabla_X J}{J}\sin\theta\right)Y + \cos\theta(\frac{k'}{J})Y \\ &\quad + \sin\theta\left(\frac{\nabla_Y J}{J}\cos\theta - \frac{\nabla_X J}{J}\sin\theta\right)X \\ &= \left(\frac{k'}{J} + \frac{\nabla_X J}{J}\sin\theta - \frac{\nabla_Y J}{J}\cos\theta\right)N,\end{aligned}$$

and thus,

$$k = \left(\frac{k'}{J} + \frac{\nabla_X J}{J}\sin\theta - \frac{\nabla_Y J}{J}\cos\theta\right)$$

$$= \frac{k'}{J} - \frac{1}{J}\nabla_N J.$$

That is,

$$k' = kJ + \nabla_N J. \tag{4}$$

J is bounded away from 0 and both J and $\nabla_N J$ are bounded. So $\lim_{t\to\omega} |k(p,t)|$ is unbounded if and only if $\lim_{t\to\omega} |k'(p,t)|$ is also unbounded.

When $V = k$, i.e., for the curvature flow in a surface M, we have

$$V' = \frac{1}{J}V = \frac{k}{J} = \frac{k'}{J^2} - \frac{\nabla_N J}{J^2}.$$

So the curvature flow in a surface corresponds to the following flow in $\mathbb{R}^2$:

$$\frac{\partial \gamma'}{\partial t} = \left(\frac{k'}{J^2} - \frac{\nabla_N J}{J^2}\right) N'. \tag{1}$$

3. Monotonicity of an Isoperimetric Ratio

From [16], when a closed curve evolves under the curvature flow in a surface, the solution remains smooth and embedded as long as its curvature remains bounded. If a singularity develops in finite time, then the curve shrinks to a point. So when t is close enough to the blow-up time ω, we may assume that the curve is contained in a small neighborhood of the collapsing point on the surface. Now by Theorem 2.3, using a local conformal diffeomorphism $\phi : U(\subseteq M) \to U' \subseteq \mathbb{R}^2$ between compact neighborhoods, we get a corresponding flow in the plane which satisfies the following equation:

$$\frac{\partial \gamma'}{\partial t} = \left(\frac{k'}{J^2} - \frac{\nabla_N J}{J^2}\right) N', \tag{1}$$

where $\gamma'(p,t) = \phi(\gamma(p,t))$, k' is the curvature of γ' in U', and N' is the unit normal vector. Our next goal is the following Lemma:

Main Lemma. *If $\gamma'(\cdot,t)$ is evolving by the parabolic flow (1), t_0 is close enough to the blow-up time $\omega < \infty$, and $\overline{G} < \frac{\pi}{2}$, then there is some $\varepsilon > 0$ such that $\overline{G}(\gamma'(\cdot,t)) > \epsilon$ for all $t \in [t_0, \omega)$.*

Hamilton [18] showed that the isoperimetric ratio $\overline{G}(\gamma(\cdot,t))$ improves under the curvature flow in the plane when $\overline{G}(\gamma(\cdot,t)) \le \pi$. We prove the main lemma by showing that the isoperimetric ratio $\overline{G}(\gamma'(\cdot,t))$ improves under the parabolic flow (1).

For a smooth embedded closed curve γ in $\mathbb{R}^2$, consider any curve Γ which divides the region enclosed by γ into two pieces with areas A_1 and A_2, where

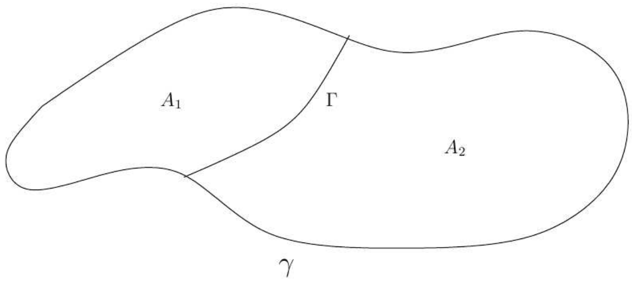

Figure 1

$A_1 + A_2 = A$ is the area enclosed by γ. Let L be the length of Γ. Define the ratio

$$G(\gamma, \Gamma) = L^2 \left(\frac{1}{A_1} + \frac{1}{A_2} \right), \tag{5}$$

and let

$$\overline{G}(\gamma) = \inf_{\Gamma} G(\gamma, \Gamma) \tag{6}$$

be the least possible value of $G(\gamma, \Gamma)$ for all curve segments Γ. Hamilton [18] takes the infimum over all possible straight lines for (6). He also defines another isoperimetric ratio, and for that ratio he takes the infimum over all possible curves. We will use the following theorem of Hamilton.

Lemma 3.1. (see [18]) *The minimum $\overline{G}(\gamma)$ is attained by a single smooth curve Γ_0 of constant curvature perpendicular to γ.*

We now show, in the rest of this section, that the isoperimetric ratio $\overline{G}(\gamma'(\cdot, t))$ improves under the parabolic flow (1).

Let's fix the time at $t = t_0$, and consider any one-parameter family of curves Γ_μ with parameter $\mu \in [-\mu_0, \mu_0]$, $\mu_0 > 0$. We will compute the first and second variation of the length $L(\Gamma_\mu)$ and the areas $A_1(\Gamma_\mu)$ and $A_2(\Gamma_\mu)$. We assume Γ_0 is our arc of constant curvature which gives the infimum for $G(\gamma'(\cdot, t_0), \Gamma_\mu)$ at $\mu = 0$.

In polar coordinates, Γ_μ is given by the graph of

$$r = r(\theta, \mu), \quad \theta \in [\theta_-(\mu), \theta_+(\mu)],$$

where θ_- is the portion of γ' near where it meets the bottom of Γ_0 and θ_+ is

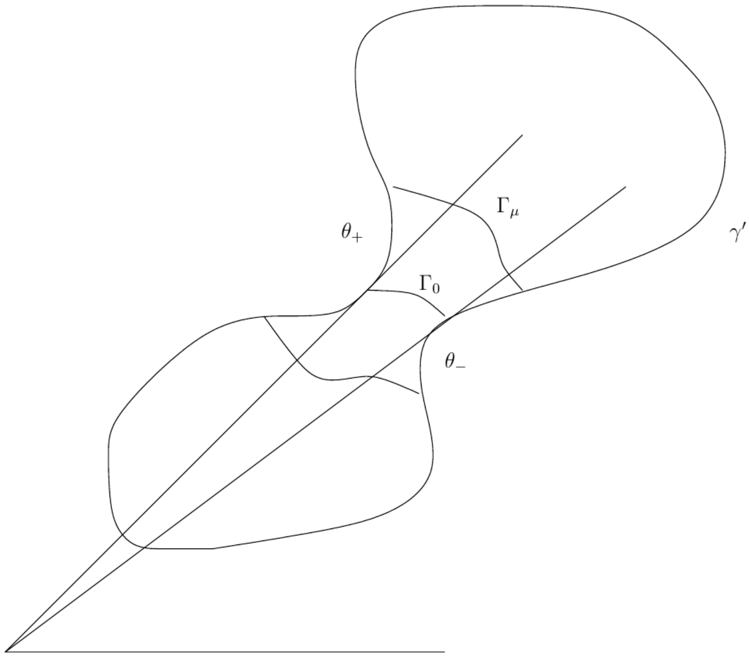

Figure 2

the portion of γ' near where it meets the top of Γ_0. So,

$$\Gamma_0 = \{(r_0, \theta) : r_0 = \frac{1}{K_0}, \theta \in [\theta_-(0), \theta_+(0)]\},$$

and we have

$$\left.\frac{\partial r}{\partial \theta}\right|_{\mu=0} = 0, \quad \text{and} \quad \left.\frac{\partial^2 r}{\partial \theta^2}\right|_{\mu=0} = 0. \tag{7}$$

Since Γ_0 is perpendicular to $\gamma'(\cdot, t_0)$ at $\mu = 0$, and we have

$$\left.\frac{\partial \theta_+}{\partial \mu}\right|_{\mu=0} = 0, \quad \text{and} \quad \left.\frac{\partial \theta_-}{\partial \mu}\right|_{\mu=0} = 0. \tag{8}$$

The curve γ' has curvatures k'_+ at $\theta_+(0)$ and k'_- at $\theta_-(0)$ which can be computed as the curvatures of the graphs of

$$\theta_+(\mu), \quad r_+(\mu) = r(\theta_+(\mu), \mu)$$

and

$$\theta_-(\mu), \quad r_-(\mu) = r(\theta_-(\mu), \mu).$$

The curvature of a parameterized curve $P(\mu) = (r\cos\theta, r\sin\theta)$ is given by

$$k = \frac{|P'(\mu) \times P''(\mu)|}{|P'(\mu)|^3}.$$

By using the formulas above, we get

$$r_0 \frac{d^2\theta_+}{d\mu^2}\bigg|_{\mu=0} = -k_+^{'} \left(\frac{\partial r_+}{\partial \mu}\bigg|_{\mu=0} \right)^2, \tag{9}$$

and

$$r_0 \frac{d^2\theta_-}{d\mu^2}\bigg|_{\mu=0} = k_+^{'} \left(\frac{\partial r_-}{\partial \mu}\bigg|_{\mu=0} \right)^2. \tag{10}$$

For the variation, let the velocity $v = \frac{\partial r}{\partial \mu}\Big|_{\mu=0}$ and the acceleration $z = \frac{\partial^2 r}{\partial \mu^2}\Big|_{\mu=0}$.

The arclength is given by

$$L(\mu) = L(\Gamma_\mu) = \int_{\theta_-(\mu)}^{\theta_+(\mu)} \sqrt{r^2 + \left(\frac{\partial r}{\partial \theta}\right)^2}\, d\theta.$$

Therefore,

$$\frac{dL}{d\mu} = \int_{\theta_-(\mu)}^{\theta_+(\mu)} \frac{1}{2}\left(r^2 + \left(\frac{\partial r}{\partial \theta}\right)^2\right)^{-1/2} \left(2r\frac{\partial r}{\partial \mu} + 2\frac{\partial r}{\partial \theta}\frac{\partial^2 r}{\partial \theta \partial \mu}\right) d\theta$$
$$+ \sqrt{r^2 + \left(\frac{\partial r}{\partial \theta}\right)^2}\,\Bigg|_{\theta_+(\mu)} \frac{\partial \theta_+}{\partial \mu} - \sqrt{r^2 + \left(\frac{\partial r}{\partial \theta}\right)^2}\,\Bigg|_{\theta_-(\mu)} \frac{\partial \theta_-}{\partial \mu}.$$

Thus,

$$\frac{dL}{d\mu}\bigg|_{\mu=0} = \int_{\theta_-(0)}^{\theta_+(0)} v\, d\theta. \tag{11}$$

Now consider the second variation of L.

$$\frac{d^2L}{d\mu^2} = \int_{\theta_-(\mu)}^{\theta_+(\mu)} \Bigg[-\frac{1}{2}\left(r^2 + \left(\frac{\partial r}{\partial \theta}\right)^2\right)^{-3/2} 2\left(r\frac{\partial r}{\partial \mu} + \frac{\partial r}{\partial \theta}\frac{\partial^2 r}{\partial \theta \partial \mu}\right)^2$$
$$+ \left(r^2 + \left(\frac{\partial r}{\partial \theta}\right)^2\right)^{-1/2} \left(\left(\frac{\partial r}{\partial \mu}\right)^2 + r\frac{\partial^2 r}{\partial \mu^2} + \left(\frac{\partial^2 r}{\partial \theta \partial \mu}\right)^2 + \frac{\partial r}{\partial \theta}\frac{\partial^3 r}{\partial \theta \partial \mu^2}\right) \Bigg] d\theta$$

$$+ \sqrt{r^2 + \left(\frac{\partial r}{\partial \theta}\right)^2}\Bigg|_{\theta_+(\mu)} \frac{\partial^2 \theta_+}{\partial \mu^2} + \frac{d}{d\mu}\left(\sqrt{r^2 + \left(\frac{\partial r}{\partial \theta}\right)^2}\Bigg|_{\theta_+(\mu)}\right)\frac{\partial \theta_+}{\partial \mu}$$
$$+ \sqrt{r^2 + \left(\frac{\partial r}{\partial \theta}\right)^2}\Bigg|_{\theta_-(\mu)} \frac{\partial^2 \theta_-}{\partial \mu^2} + \frac{d}{d\mu}\left(\sqrt{r^2 + \left(\frac{\partial r}{\partial \theta}\right)^2}\Bigg|_{\theta_-(\mu)}\right)\frac{\partial \theta_-}{\partial \mu}.$$

Thus,

$$\frac{d^2 L}{d\mu^2}\bigg|_{\mu=0} = \int_{\theta_-(0)}^{\theta_+(0)} z\, d\theta + K_0 \int_{\theta_-(0)}^{\theta_+(0)} \left(\frac{dv}{d\theta}\right)^2 d\theta - (k_+' v_+^2 + k_-' v_-^2). \tag{12}$$

Now we compute the first and second variation of the areas A_1 and A_2. Since $A_1(\mu) + A_2(\mu) = A$, we have $\frac{dA_1}{d\mu} = -\frac{dA_2}{d\mu}$ and $\frac{d^2 A_1}{d\mu^2} = -\frac{d^2 A_2}{d\mu^2}$. We have

$$\text{Area} \quad A = \int\int r dr d\theta = \int\int r \frac{\partial r}{\partial \mu} d\mu d\theta.$$

If $A_1(\mu)$ denotes the area on the origin side of Γ_μ, then

$$A_1(\mu) - A_1(0) = \int_0^\mu \int_{\theta_-(\tau)}^{\theta_+(\tau)} r \frac{\partial r}{\partial \tau} d\theta d\tau.$$

Therefore,

$$\frac{dA_1}{d\mu} = \int_{\theta_-(\mu)}^{\theta_+(\mu)} r \frac{\partial r}{\partial \mu}\, d\theta \quad \Rightarrow \quad \frac{dA_1}{d\mu}\bigg|_{\mu=0} = \frac{1}{K_0}\int_{\theta_-(0)}^{\theta_+(0)} v\, d\theta,$$

and we have

$$\frac{dA_1}{d\mu}\bigg|_{\mu=0} = -\frac{dA_2}{d\mu}\bigg|_{\mu=0} = \frac{1}{K_0}\int_{\theta_-(0)}^{\theta_+(0)} v\, d\theta. \tag{13}$$

Now,

$$\frac{d^2 A_1}{d\mu^2} = \int_{\theta_-(\mu)}^{\theta_+(\mu)} r\frac{\partial^2 r}{\partial \mu^2} + \left(\frac{\partial r}{\partial \mu}\right)^2 d\theta + r\frac{\partial r}{\partial \mu}\bigg|_{\theta_+(\mu)} \frac{\partial \theta_+}{\partial \mu} - r\frac{\partial r}{\partial \mu}\bigg|_{\theta_-(\mu)} \frac{\partial \theta_-}{\partial \mu}.$$

Thus,

$$\frac{d^2 A_1}{d\mu^2}\bigg|_{\mu=0} = -\frac{d^2 A_2}{d\mu^2}\bigg|_{\mu=0} = \frac{1}{K_0}\int_{\theta_-(0)}^{\theta_+(0)} z\, d\theta + \int_{\theta_-(0)}^{\theta_+(0)} v^2\, d\theta. \tag{14}$$

Having found the first and second variations of L, A_1 and A_2, we can now write down the condition that $G = L^2\left(\frac{1}{A_1} + \frac{1}{A_2}\right)$ attains its minimum at Γ_0. As usual, this says that $\frac{dG}{d\mu}\Big|_{\mu=0} = 0$ and $\frac{d^2 G}{d\mu^2}\Big|_{\mu=0} \geq 0$. It is easier to express this inequality in terms of logarithms; thus

$$G = L^2\left(\frac{1}{A_1}+\frac{1}{A_2}\right) = L^2\left(\frac{A}{A_1A_2}\right) \Rightarrow \ln G = 2\ln L + \ln A - \ln A_1 - \ln A_2.$$

Since

$$0 = \frac{d}{d\mu}\bigg|_{\mu=0}\ln G = \frac{2}{L}\,\frac{dL}{d\mu}\bigg|_{\mu=0} + \frac{1}{A}\,\frac{dA}{d\mu}\bigg|_{\mu=0} - \frac{1}{A_1}\,\frac{dA_1}{d\mu}\bigg|_{\mu=0} - \frac{1}{A_2}\,\frac{dA_2}{d\mu}\bigg|_{\mu=0},$$

we have

$$\frac{2}{L}\int_{\theta_-(0)}^{\theta_+(0)} v\,d\theta - \frac{1}{A_1}\frac{1}{K_0}\int_{\theta_-(0)}^{\theta_+(0)} v\,d\theta + \frac{1}{A_2}\frac{1}{K_0}\int_{\theta_-(0)}^{\theta_+(0)} v\,d\theta = 0.$$

Therefore,

$$\frac{2K_0}{L} = \frac{1}{A_1} - \frac{1}{A_2}. \tag{15}$$

Next, we have

$$\begin{aligned}
0 \le \frac{d^2}{d\mu^2}\bigg|_{\mu=0}\ln G &= \frac{d}{d\mu}\bigg|_{\mu=0}\left(\frac{2}{L}\frac{dL}{d\mu} - \frac{1}{A_1}\frac{dA_1}{d\mu} - \frac{1}{A_2}\frac{dA_2}{d\mu}\right)\\
&= \frac{d}{d\mu}\bigg|_{\mu=0}\left(\frac{2}{L}\frac{dL}{d\mu} - \frac{dA_1}{d\mu}\left(\frac{1}{A_1}-\frac{1}{A_2}\right)\right)
\end{aligned}$$

$$= \left[-\frac{2}{L^2}\left(\frac{dL}{d\mu}\right)^2 + \frac{2}{L}\frac{d^2L}{d\mu^2} - \frac{d^2A_1}{d\mu^2}\left(\frac{1}{A_1}-\frac{1}{A_2}\right) + \frac{dA_1}{d\mu}\left(\frac{1}{A_1^2}\frac{dA_1}{d\mu} - \frac{1}{A_2^2}\frac{dA_2}{d\mu}\right)\right]_{\mu=0}$$

$$= -\frac{2}{L^2}\left(\int_{\theta_-(0)}^{\theta_+(0)} v d\theta\right)^2 + \frac{2}{L}\left[\int_{\theta_-(0)}^{\theta_+(0)} z d\theta + K_0\int_{\theta_-(0)}^{\theta_+(0)}\left(\frac{dv}{d\theta}\right)^2 d\theta - (k_+^{'}v_+^2 + k_-^{'}v_-^2)\right]$$

$$-\left(\frac{1}{A_1}-\frac{1}{A_2}\right)\left[\frac{1}{K_0}\int_{\theta_-(0)}^{\theta_+(0)} z d\theta + \int_{\theta_-(0)}^{\theta_+(0)} v^2 d\theta\right] + \left(\frac{1}{A_1^2}+\frac{1}{A_2^2}\right)\left[\frac{1}{K_0}\int_{\theta_-(0)}^{\theta_+(0)} v d\theta\right]^2$$

$$= \left[-\frac{2}{L^2} + \frac{1}{K_0^2}\left(\frac{1}{A_1^2}+\frac{1}{A_2^2}\right)\right]\left(\int_{\theta_-(0)}^{\theta_+(0)} v d\theta\right)^2 + \left[\frac{2}{L} - \frac{1}{K_0}\left(\frac{1}{A_1}-\frac{1}{A_2}\right)\right]\int_{\theta_-(0)}^{\theta_+(0)} z d\theta$$

$$+ \frac{2K_0}{L}\int_{\theta_-(0)}^{\theta_+(0)}\left(\frac{dv}{d\theta}\right)^2 d\theta - \frac{2}{L}(k_+^{'}v_+^2 + k_-^{'}v_-^2) - \left(\frac{1}{A_1}-\frac{1}{A_2}\right)\int_{\theta_-(0)}^{\theta_+(0)} v^2\,d\theta.$$

So, by using (15), we get

$$\frac{2}{L}(k_+^{'}v_+^2 + k_-^{'}v_-^2) \le \frac{1}{2K_0^2}\left(\frac{1}{A_1}+\frac{1}{A_2}\right)^2\left(\int_{\theta_-(0)}^{\theta_+(0)} v\,d\theta\right)^2$$

$$+\frac{2K_0}{L}\int_{\theta_-(0)}^{\theta_+(0)}\left(\left(\frac{dv}{d\theta}\right)^2 - v^2\right) d\theta. \quad (16)$$

The curvature flow in a surface corresponds to the following flow in the plane $\mathbb{R}^2$,

$$\frac{\partial\gamma'}{\partial t} = \left(\frac{k'}{J^2} - \frac{\nabla_N J}{J^2}\right) N' = V'N'.$$

We will now use (15) and (16) to show that $\frac{d}{dt}\big|_{t=t_0} \ln G > 0$. First we need to compute the evolution of L, A, A_1 and A_2 at time t_0. The evolution of the length L is the sum of the normal velocity of $\gamma'(\cdot, t)$ at the two ends of Γ_0, so that

$$\left.\frac{dL}{dt}\right|_{t=t_0} = -\left(V'_+ + V'_-\right).$$

The evolution of the areas are given by:

$$\left.\frac{dA}{dt}\right|_{t=t_0} = -\int_{\gamma'(\cdot,t_0)} V'\,ds, \quad \left.\frac{dA_1}{dt}\right|_{t=t_0} = -\int_{\gamma'_1(\cdot,t_0)} V'\,ds,$$

$$\left.\frac{dA_2}{dt}\right|_{t=t_0} = -\int_{\gamma'_2(\cdot,t_0)} V'\,ds.$$

Since

$$G = L^2\left(\frac{1}{A_1} + \frac{1}{A_2}\right) = L^2\frac{A}{A_1A_2},$$

we have

$$\ln G = 2\ln L + \ln A - \ln A_1 - \ln A_2,$$

and

$$\left.\frac{d}{dt}\right|_{t=t_0} \ln G = \frac{2}{L}\left.\frac{dL}{dt}\right|_{t=t_0} + \frac{1}{A}\left.\frac{dA}{dt}\right|_{t=t_0} - \frac{1}{A_1}\left.\frac{dA_1}{dt}\right|_{t=t_0} - \frac{1}{A_2}\left.\frac{dA_2}{dt}\right|_{t=t_0}.$$

Thus,

$$\left.\frac{d}{dt}\right|_{t=t_0} \ln G = -\frac{2}{L}\left(\frac{k'_+}{J_+^2} + \frac{k'_-}{J_-^2}\right) + \frac{2}{L}\left(\left.\frac{\nabla_N J}{J^2}\right|_+ + \left.\frac{\nabla_N J}{J^2}\right|_-\right)$$
$$-\frac{1}{A}\int_{\gamma'(\cdot,t_0)} V'\,ds + \frac{1}{A_1}\int_{\gamma'_1(\cdot,t_0)} V'\,ds + \frac{1}{A_2}\int_{\gamma'_2(\cdot,t_0)} V'\,ds. \quad (17)$$

If we choose the variation such that

$$v(r_0, \theta) = \frac{1}{J(r_0\cos\theta, r_0\sin\theta)} \quad \theta \in [\theta_-(0), \theta_+(0)],$$

then we could use the result from (16) in (17). Consider the RHS of (16). First,

$$\int_{\theta_-(0)}^{\theta_+(0)} v\, d\theta \le \max_{\theta_-(0)\le\theta\le\theta_+(0)} \left(\frac{1}{J}\right)(\theta_+ - \theta_-).$$

Using $(\theta_+ - \theta_-) = LK_0$, we get that the first term of the RHS of (16) is bounded by

$$\frac{1}{2K_0^2}\left(\frac{1}{A_1}+\frac{1}{A_2}\right)^2 \left(\int_{\theta_-(0)}^{\theta_+(0)} v\, d\theta\right)^2 \le \frac{C_1 L^2}{2}\left(\frac{1}{A_1}+\frac{1}{A_2}\right)^2, \tag{18}$$

where

$$C_1 = \max_{\theta_-\le\theta\le\theta_+}\left(\frac{1}{J^2}\right). \tag{19}$$

Now considering the second term in the RHS of (16), we have

$$\begin{aligned} \frac{dv}{d\theta} &= -\frac{1}{J^2}[J_x(-r_0\sin\theta) + J_y(r_0\cos\theta)] \\ &\le \frac{r_0}{J^2}\sqrt{J_x^2+J_y^2}, \end{aligned}$$

so we get the bound

$$\frac{2K_0}{L}\int_{\theta_-(0)}^{\theta_+(0)} \left(\frac{dv}{d\theta}\right)^2 d\theta \le \frac{2K_0}{L}\frac{C_2}{K_0^2}(LK_0) = 2C_2, \tag{20}$$

where

$$C_2 = \max_{\theta_-\le\theta\le\theta_+}\left(\frac{J_x^2+J_y^2}{J^4}\right). \tag{21}$$

Now using (15) we get the bound for the third term in the RHS of (16),

$$\frac{2K_0}{L}\int_{\theta_-(0)}^{\theta_+(0)} v^2\, d\theta \ge \frac{2K_0}{L}C_3(LK_0) = 2C_3K_0^2 = \frac{C_3L^2}{2}\left(\frac{1}{A_1}-\frac{1}{A_2}\right)^2, \tag{22}$$

where

$$C_3 = \min_{\theta_-\le\theta\le\theta_+}\left(\frac{1}{J^2}\right). \tag{23}$$

So now (16), (18), (20), and (22) give

$$\frac{2}{L}\left(\frac{k'_+}{J_+^2}+\frac{k'_-}{J_-^2}\right) \le \frac{C_1L^2}{2}\left(\frac{1}{A_1}+\frac{1}{A_2}\right)^2 + 2C_2 - \frac{C_3L^2}{2}\left(\frac{1}{A_1}-\frac{1}{A_2}\right)^2. \tag{24}$$

So we have a bound on the first term in the RHS of (17). We now bound the last three terms in the RHS of (17). First,

$$\int_{\gamma'} V'\, ds = \int_{\gamma'}\left(\frac{k'}{J^2}-\frac{\nabla_N J}{J^2}\right) ds \ge C_4\int_{\gamma'} k'\, ds - C_5\int_{\gamma'} ds,$$

where

$$C_4 = \min_{U'} \left(\frac{1}{J^2} \right), \tag{25}$$

and

$$C_5 = \max_{U'} \left(\frac{\nabla_N J}{J^2} \right). \tag{26}$$

Therefore,

$$\begin{aligned}
&-\frac{1}{A}\int_{\gamma'(\cdot,t_0)} V'\,ds + \frac{1}{A_1}\int_{\gamma_1'(\cdot,t_0)} V'\,ds + \frac{1}{A_2}\int_{\gamma_2'(\cdot,t_0)} V'\,ds \\
&= \left(\frac{1}{A_1} - \frac{1}{A}\right)\int_{\gamma_1'} \left(\frac{k'}{J^2} - \frac{\nabla_N J}{J^2}\right) ds + \left(\frac{1}{A_2} - \frac{1}{A}\right)\int_{\gamma_2'} \left(\frac{k'}{J^2} - \frac{\nabla_N J}{J^2}\right) ds \\
&\geq \left(\frac{1}{A_1} - \frac{1}{A}\right)\left[C_4\int_{\gamma_1'} k'\,ds - C_5\int_{\gamma_1'} ds\right] + \left(\frac{1}{A_2} - \frac{1}{A}\right)\left[C_4\int_{\gamma_2'} k'\,ds - C_5\int_{\gamma_2'} ds\right] \\
&= \left(\frac{1}{A_1} - \frac{1}{A}\right)\left[C_4(\pi - (\theta_+ - \theta_-)) - C_5 L(\gamma_1')\right] \\
&\qquad + \left(\frac{1}{A_2} - \frac{1}{A}\right)\left[C_4(\pi + (\theta_+ - \theta_-)) - C_5 L(\gamma_2')\right] \\
&= \frac{C_4\pi(A_1^2 + A_2^2)}{A_1A_2(A_1 + A_2)} + \frac{C_4 L^2}{2}\left(\frac{1}{A_1} - \frac{1}{A_2}\right)\frac{(A_1^2 - A_2^2)}{A_1A_2(A_1 + A_2)} \\
&\qquad + C_5\left[\frac{L(\gamma')}{A} - \frac{L(\gamma_1')}{A_1} - \frac{L(\gamma_2')}{A_2}\right].
\end{aligned}$$

Since

$$\frac{1}{2}(A_1 + A_2)^2 \leq (A_1^2 + A_2^2) \leq (A_1 + A_2)^2,$$

we have

$$\frac{(A_1 + A_2)^2}{2A_1A_2(A_1 + A_2)} \leq \frac{(A_1^2 + A_2^2)}{A_1A_2(A_1 + A_2)} \leq \frac{(A_1 + A_2)^2}{A_1A_2(A_1 + A_2)};$$

that is,

$$\frac{1}{2}\left(\frac{1}{A_1} + \frac{1}{A_2}\right) \leq \frac{(A_1^2 + A_2^2)}{A_1A_2(A_1 + A_2)} \leq \left(\frac{1}{A_1} + \frac{1}{A_2}\right).$$

Also,

$$\frac{L(\gamma')}{A} - \frac{L(\gamma_1')}{A_1} - \frac{L(\gamma_2')}{A_2} = \frac{L_1 + L_2}{A_1 + A_2} - \frac{L_1}{A_1} - \frac{L_2}{A_2} \geq -L'\left(\frac{1}{A_1} + \frac{1}{A_2}\right),$$

where $L' = \max(L_1, L_2)$.

So now we have bound on the last three terms in the RHS of (17):

$$-\frac{1}{A}\int_{\gamma'(\cdot,t_0)} V'\,ds + \frac{1}{A_1}\int_{\gamma_1'(\cdot,t_0)} V'\,ds + \frac{1}{A_2}\int_{\gamma_2'(\cdot,t_0)} V'\,ds$$
$$\geq \frac{C_4\pi}{2}\left(\frac{1}{A_1}+\frac{1}{A_2}\right) - \frac{C_4L^2}{2}\left(\frac{1}{A_1}-\frac{1}{A_2}\right)^2 - C_5L'\left(\frac{1}{A_1}+\frac{1}{A_2}\right). \quad (27)$$

Next we compute the second term in the RHS of (17). Let

$$\frac{\nabla_N J}{J^2}(\theta) = \frac{\nabla_N J}{J^2}(r_0\cos\theta, r_0\sin\theta, \theta).$$

Then

$$\left.\frac{\nabla_N J}{J^2}\right|_- = \frac{\nabla_N J}{J^2}(r_0\cos\theta_-, r_0\sin\theta_-, \theta_-) = \frac{\nabla_N J}{J^2}(\theta_-),$$

and

$$\left.\frac{\nabla_N J}{J^2}\right|_+ = \frac{\nabla_N J}{J^2}(r_0\cos\theta_+, r_0\sin\theta_+, \theta_+ + \pi) = -\frac{\nabla_N J}{J^2}(\theta_+).$$

By the mean value theorem,

$$\left(\left.\frac{\nabla_N J}{J^2}\right|_+ + \left.\frac{\nabla_N J}{J^2}\right|_-\right) = \left(\frac{\nabla_N J}{J^2}\right)'(\theta_0)(\theta_- - \theta_+),$$

for some $\theta_0 \in (\theta_-, \theta_+)$. Therefore

$$\frac{2}{L}\left(\left.\frac{\nabla_N J}{J^2}\right|_+ + \left.\frac{\nabla_N J}{J^2}\right|_-\right) = -\frac{2}{L}\left(\frac{\nabla_N J}{J^2}\right)'(\theta_0)(LK_0)$$
$$= -\left(\frac{\nabla_N J}{J^2}\right)'(\theta_0)L\left(\frac{1}{A_1}-\frac{1}{A_2}\right). \quad (28)$$

Thus (17), (24), (27), and (28) give

$$\left.\frac{d}{dt}\right|_{t=t_0} \ln G \geq -\frac{C_1L^2}{2}\left(\frac{1}{A_1}+\frac{1}{A_2}\right)^2 - 2C_2 + \frac{C_3L^2}{2}\left(\frac{1}{A_1}-\frac{1}{A_2}\right)^2$$
$$+\frac{C_4\pi}{2}\left(\frac{1}{A_1}+\frac{1}{A_2}\right) - \frac{C_4L^2}{2}\left(\frac{1}{A_1}-\frac{1}{A_2}\right)^2 - C_5L'\left(\frac{1}{A_1}+\frac{1}{A_2}\right)$$
$$-\left(\frac{\nabla_N J}{J^2}\right)'(\theta_0)L\left(\frac{1}{A_1}-\frac{1}{A_2}\right)$$
$$= \frac{1}{2}\left(\frac{1}{A_1}+\frac{1}{A_2}\right)\left(C_4\pi - C_1\overline{G} - 2C_5L' - 2\left(\frac{\nabla_N J}{J^2}\right)'(\theta_0)\,\frac{A_2-A_1}{A_1+A_2}\,L\right)$$

$$+\frac{L^2}{2}\left(\frac{1}{A_1}-\frac{1}{A_2}\right)^2(C_3-C_4)-2C_2. \quad (29)$$

If t_0 is close enough to the blow-up time ω, we can make $C_4=\min_{U'}\left(\frac{1}{J^2}\right)$ and $C_1=\max_{\theta_-\le\theta\le\theta_+}\left(\frac{1}{J^2}\right)$ approach 1, and $C_5=\max_{U'}\left(\frac{\nabla_N J}{J^2}\right)$ and $C_2=\max_{\theta_-\le\theta\le\theta_+}\left(\frac{J_x^2+J_y^2}{J^4}\right)$ approach 0. The term $\left(\frac{\nabla_N J}{J^2}\right)'(\theta_0)\frac{A_2-A_1}{A_1+A_2}$ is bounded. We also have $C_3=\min_{\theta_-\le\theta\le\theta_+}\left(\frac{1}{J^2}\right)\ge C_4$. The lengths L' and L approach 0, and $\left(\frac{1}{A_1}+\frac{1}{A_2}\right)$ becomes larger. Hence when $\overline{G}$ gets smaller (say $<\frac{\pi}{2}$), $\frac{d}{dt}\Big|_{t=t_0}\ln G>0$.

This completes the proof of the main lemma..

In the next section we will study the formation of singularity by re-scaling the solutions, and then prove our main theorem using the main lemma.

4. The Limit of the Re-Scaled Solutions

If the evolution equation has a smooth solution on a maximal time interval $0\le t<\omega<\infty$, then the supremum norm of the curvature must blow up as $t\to\omega$. We say that $Q\in\mathbb{R}^2$ is a *blow-up point or singularity* if there is $p\in S^1$ such that $\gamma(p,t)\to Q$ and $k(p,t)$ becomes unbounded as $t\to\omega$. We define $\{(p_n,t_n)\in S^1\times[0,\omega)\}$ to be a (essential) *blow-up sequence* if $\lim_{n\to\infty}t_n=\omega$, $\lim_{n\to\infty}k(p_n,t_n)=\infty$, and

$$|k(p,t)|\le|k(p_n,t_n)| \quad p\in S^1,\ t\in(0,t_n].$$

Let $M_t=\sup k^2(\cdot,t)$. Then we will use the following dilation-invariant categorization of singularity formation:

Type-I singularity if $\lim_{t\to\omega}M_t.(\omega-t)$ is bounded, and

Type-II singularity if $\lim_{t\to\omega}M_t.(\omega-t)$ is unbounded.

We next re-scale the solution along a blow-up sequence $\{(p_n,t_n)\}$: for every n we obtain a new solution γ_n, from γ by translating $t_n\mapsto 0$, and dilating the solution in space and time (scaling time as space squared) so that $k_n^2(p_n,0)\mapsto 1$. First, we will give a precise definition: We have

$$\gamma: S^1\times[0,\omega)\to\mathbb{R}^2.$$

We define the re-scaled solutions γ_n of γ along the blow-up sequence $\{(p_n,t_n)\}$ to be as follows:

$$\gamma_n: S^1\times[-\lambda_n^2t_n,\lambda_n^2(\omega-t_n))\to\mathbb{R}^2$$

is given by

$$\gamma_n(\cdot,\bar{t}) = \lambda_n[\gamma(\cdot,t)] = \lambda_n[\gamma(\cdot,t_n+\lambda_n^{-2}\bar{t})],$$

where $\lambda_n = |k(p_n,t_n)|$ and $\bar{t} = \lambda_n^2(t-t_n)$. So, we have

$$t \in [0,\omega) \Leftrightarrow \bar{t} \in [-\lambda_n^2 t_n, \lambda_n^2(\omega - t_n)) = [a_n,\omega_n)\,\text{say}.$$

That is,

$$\gamma_n : S^1 \times [a_n,\omega_n) \to \mathbb{R}^2$$

is given by

$$\gamma_n(\cdot,\bar{t}) = \lambda_n\gamma(\cdot,t).$$

Since $\lambda_n \to \infty$, we have $\lim_{n\to\infty} a_n = -\infty$ and

$$\lim_{n\to\infty}\omega_n = \begin{cases} \text{finite} & \text{if type I,} \\ +\infty & \text{if type II.} \end{cases}$$

The curvature of γ_n satisfies $|k_n(p,\bar{t})| \le 1 \quad$ for all $\quad \bar{t} \in [a_n,0]$. We have

$$\frac{\partial\gamma_n}{\partial\bar{t}} = \lambda_n \frac{\partial\gamma}{\partial t}\frac{dt}{d\bar{t}} = \lambda_n\frac{\partial\gamma}{\partial t}(\lambda_n^{-2}).$$

So

$$\frac{\partial\gamma_n}{\partial\bar{t}} = \frac{1}{\lambda_n}\left(\frac{\partial\gamma}{\partial t}\right).$$

Since

$$\frac{\partial\gamma}{\partial t} = V(T,k)N = \left(\frac{k}{J^2} - \frac{\nabla_N J}{J^2}\right)N$$

(notice that we have dropped the prime), we have

$$\begin{aligned}\frac{\partial\gamma_n}{\partial\bar{t}} &= \frac{1}{\lambda_n}V(T,\lambda_n k_n)N \\ &= \left(\frac{k_n}{J^2} - \frac{1}{\lambda_n}\frac{\nabla_N J}{J^2}\right)N.\end{aligned}$$

A limit solution, if it exists, may be a family of noncompact curves. So think of our solutions as a family of $L(t)$ (length of $\gamma(\cdot,t)$) periodic curves,

$$\widetilde{\gamma_n} : \mathbb{R}\times[a_n,\omega_n) \to \mathbb{R}^2,$$

such that $\widetilde{\gamma_n}(0,\cdot) = \gamma_n(p_n,\cdot)$. We also parameterize the curves by arclength from the origin $0 \in \mathbb{R}$.

Now as in [4], a uniform bound on the curvature implies bounds on the higher derivatives. Therefore, by the Ascoli-Arzela Theorem one can extract a subsequence of $\widetilde{\gamma_n}(\cdot,t)$ which converges on compact sets of $\mathbb{R}\times(-\infty,\omega_\infty)$ to a smooth family of curves $\widetilde{\gamma_\infty}$.

The limit solution $\widetilde{\gamma_\infty}$ is either closed, or unbounded and complete. We will denote by γ_∞ one period, possibly infinite, of $\widetilde{\gamma_\infty}$, which satisfies

$$\frac{\partial \gamma_\infty}{\partial \bar{t}} = \frac{k_\infty}{J^2(Q)} N = k_\infty N,$$

where Q is the collapsing point of $\gamma(\cdot, t)$, and k_∞ is the curvature of $\gamma_\infty(\cdot, \bar{t})$. So $|k_\infty(p, \bar{t})| \leq 1$ for all $\bar{t} \leq 0$ with $|k_\infty(0,0)| = 1$, and hence the process of re-scaling does not allow the limit solution to be trivial, that is, a straight line.

Lemma 4.1. (see [2]) *For a closed embedded curve in the plane evolving by curvature flow we have*

$$\frac{d}{dt} \int_{\gamma(\cdot,t)} |k|\, ds = -2 \sum_{p:k(p,t)=0} \left| \frac{\partial k}{\partial s} \right|.$$

Theorem 4.1. *γ_∞ is a family of convex curves.*

Proof. On the limit solution, $\int_{\gamma_\infty(\cdot,t)} |k_\infty|\, ds$ is constant. We also have

$$\frac{d}{dt} \int_{\gamma_\infty(\cdot,t)} |k_\infty|\, ds = -2 \sum_{p:k_\infty(p,t)=0} \left| \frac{\partial k_\infty}{\partial s} \right|.$$

Hence,

$$\int_{-\infty}^{\omega_\infty} \sum_{p:k_\infty(p,t)=0} \left| \frac{\partial k_\infty}{\partial s} \right| dt = 0.$$

Therefore, any inflection points for the limit curve must be degenerate (i.e., $k_\infty = \frac{\partial k_\infty}{\partial s} = 0$). So [6] implies that if a solution has degenerate inflection points for any interval in time, then the solution must be a line. Since γ_∞ is not trivial, the family of curves must have no inflection points and therefore must all be convex. □

4.1. Limiting Shapes of Re-Scaled Solutions along Blow-Up Sequences

4.1.1. Type-I Singularities

In this section we assume $\{(p_n, t_n)\}$ is type-I blow-up sequence. We will prove that the re-scaled solutions γ_∞ on $[0, \omega_\infty)$ of the curvature flow converge to a solution which moves simply by homothety. It is convenient to drop the ∞ symbol in this section and consider the solution as $\gamma(p, t)$ on $[0, \omega)$.

Now we want to re-scale $\gamma(\cdot, t)$ near a singular point as $t \to \omega$, such that the re-scaled curve remains uniformly bounded. So we define the re-scaled solution

$\overline{\gamma}$ of the solution γ on $[0,\omega)$ by

$$\overline{\gamma}(p,\overline{t}\,)=\frac{\gamma(p,t)}{\sqrt{2(\omega-t)}},$$

where

$$\overline{t}=-\frac{1}{2}\ln(\omega-t)\in\left[-\frac{1}{2}\ln\omega,+\infty\right)\equiv[t_0,+\infty).$$

That is,

$$\overline{\gamma}:S^1\times[t_0,\infty)\to\mathbb{R}^2$$

and

$$\frac{d\overline{t}}{dt}=\frac{1}{2(\omega-t)}\Rightarrow\frac{\partial}{\partial\overline{t}}=2(\omega-t)\frac{\partial}{\partial t}.$$

Arclength is given by

$$\overline{s}(p,\overline{t})=\int_0^p\left|\frac{\partial\overline{\gamma}}{\partial q}(q,\overline{t})\right|\,dq.$$

Differentiating,

$$\overline{v}(p,t)=\frac{\partial\overline{s}}{\partial p}(p,\overline{t})=\left|\frac{\partial\overline{\gamma}}{\partial p}(p,\overline{t})\right|=\frac{1}{\sqrt{2(\omega-t)}}\left|\frac{\partial\gamma}{\partial p}(p,t)\right|.$$

Thus,

$$\frac{\partial\overline{s}}{\partial p}=\frac{1}{\sqrt{2(\omega-t)}}\frac{\partial s}{\partial p}\Rightarrow\frac{\partial}{\partial\overline{s}}=\sqrt{2(\omega-t)}\frac{\partial}{\partial s}.$$

Hence we have the following operators:

$$\frac{\partial}{\partial\overline{t}}=2(\omega-t)\frac{\partial}{\partial t},\qquad\frac{\partial}{\partial\overline{s}}=\sqrt{2(\omega-t)}\frac{\partial}{\partial s}.$$

Therefore,

$$\begin{aligned}\frac{\partial\overline{\gamma}}{\partial\overline{t}}&=2(\omega-t)\frac{\partial}{\partial t}\left(\frac{\gamma}{\sqrt{2(\omega-t)}}\right)\\&=\sqrt{2(\omega-t)}\frac{\partial\gamma}{\partial t}+\frac{\gamma}{\sqrt{2(\omega-t)}}\\&=\sqrt{2(\omega-t)}\frac{\partial^2\gamma}{\partial s^2}+\overline{\gamma}\\&=\frac{\partial}{\partial\overline{s}}\left(\frac{\partial\gamma}{\partial s}\right)+\overline{\gamma}.\end{aligned}$$

But,

$$\frac{\partial\overline{\gamma}}{\partial\overline{s}}=\sqrt{2(\omega-t)}\frac{\partial}{\partial s}\left(\frac{\gamma}{\sqrt{2(\omega-t)}}\right)=\sqrt{2(\omega-t)}\frac{1}{v}\frac{\partial}{\partial p}\left(\frac{\gamma}{\sqrt{2(\omega-t)}}\right)$$

$$= \sqrt{2(\omega - t)} \frac{1}{\sqrt{2(\omega - t)}} \frac{1}{v} \frac{\partial \gamma}{\partial p} = \frac{\partial \gamma}{\partial s},$$

so the re-scaled solutions satisfy the equation

$$\frac{\partial \overline{\gamma}}{\partial \overline{t}} = \frac{\partial^2 \overline{\gamma}}{\partial \overline{s}^2} + \overline{\gamma}.$$

The curvature of the modified solution is

$$\overline{k}(p, \overline{t}) = \sqrt{2(\omega - t)} k(p, t).$$

Since we are assuming the forming singularity is type-I, the curvature $\overline{k}^2(\cdot, \overline{t})$ is uniformly bounded, and all higher derivatives of the curvature are bounded as well.

Monotonicity and Self-Similar Solutions. Huisken [20] proved a general monotonicity formula for hypersurfaces moving by mean curvature flow. Then he used the monotonicity result to show that singularities satisfying the growth rate estimate $M_t \leq \frac{C}{\omega - t}$ (type-I), are asymptotically self-similar. As in [2], we apply the Huisken monotonicity formula for the curves evolving by curvature flow in a plane.

Let $\rho(x, t)$ be the backward heat kernel at $(\overrightarrow{0}, \omega)$, i.e.,

$$\rho(x, t) = \frac{1}{\sqrt{4\pi(\omega - t)}} \exp\left(\frac{-|x|^2}{4(\omega - t)} \right), \qquad t < \omega.$$

In the re-scaled setting we obtain a monotonicity formula if we define the modified kernel by

$$\overline{\rho}(x, \overline{t}) = e^{-\frac{1}{2}|x|^2}, \qquad x \in \mathbb{R}^2.$$

Theorem 4.2. (see [20]) *1. For γ, when $t \in [0, \omega)$, we have the formula*

$$\frac{d}{dt} \int_{\gamma(\cdot, t)} \rho(x, t)\, ds = -\int_{\gamma(\cdot, t)} \rho(x, t) \left| \frac{\partial^2 \gamma}{\partial s^2} + \frac{1}{2(\omega - t)} \gamma^{\perp} \right|^2 ds. \tag{30}$$

2. For $\overline{\gamma}$, when $\overline{t} \in [t_0, \infty)$, we have the formula

$$\frac{d}{d\overline{t}} \int_{\overline{\gamma}(\cdot, \overline{t})} \overline{\rho}\, d\overline{s} = -\int_{\overline{\gamma}(\cdot, \overline{t})} \overline{\rho} \left| \frac{\partial^2 \overline{\gamma}}{\partial \overline{s}^2} + \overline{\gamma}^{\perp} \right|^2 d\overline{s}, \tag{31}$$

where $\gamma^{\perp} = \gamma - \gamma^{\top}$, and $\gamma^{\top}$ is the tangential component of the position vector.

Proof. First we compute $\frac{\partial \overline{v}}{\partial \overline{t}}$. Arclength is given by

$$\overline{s}(p, \overline{t}) = \int_0^p \left| \frac{\partial \overline{\gamma}}{\partial q}(q, \overline{t}) \right| dq.$$

Differentiating,
$$\frac{\partial \overline{s}}{\partial p}(p,\overline{t}) = \left|\frac{\partial \overline{\gamma}}{\partial p}(p,\overline{t})\right| = \overline{v}(p,\overline{t}).$$
In addition,
$$\overline{v}^2 = \left\langle \frac{\partial \overline{\gamma}}{\partial p}, \frac{\partial \overline{\gamma}}{\partial p}\right\rangle,$$
which implies that
$$2\overline{v}\frac{\partial \overline{v}}{\partial \overline{t}} = 2\left\langle \frac{\partial^2 \overline{\gamma}}{\partial \overline{t}\partial p}, \frac{\partial \overline{\gamma}}{\partial p}\right\rangle.$$
Thus,
$$\begin{aligned}
\overline{v}\frac{\partial \overline{v}}{\partial \overline{t}} &= \left\langle \frac{\partial}{\partial p}(\overline{k}\,\overline{N}+\overline{\gamma}), \frac{\partial \overline{\gamma}}{\partial p}\right\rangle \quad \text{because} \quad \frac{\partial \overline{\gamma}}{\partial \overline{t}} = \frac{\partial^2 \overline{\gamma}}{\partial \overline{s}^2} + \overline{\gamma} \\
&= \left\langle \overline{k}\frac{\partial \overline{N}}{\partial p} + \frac{\partial \overline{k}}{\partial p}\overline{N} + \frac{\partial \overline{\gamma}}{\partial p}, \frac{\partial \overline{\gamma}}{\partial p}\right\rangle \\
&= \left\langle \overline{k}\left(\overline{v}(-\overline{k}\,\overline{T})\right), \frac{\partial \overline{\gamma}}{\partial p}\right\rangle + \left|\frac{\partial \overline{\gamma}}{\partial p}\right|^2 \\
&= -\overline{k}^2\overline{v}\left|\frac{\partial \overline{\gamma}}{\partial p}\right| + \left|\frac{\partial \overline{\gamma}}{\partial p}\right|^2 \\
&= -\overline{k}^2\overline{v}^2 + \overline{v}^2.
\end{aligned}$$
Hence,
$$\frac{\partial \overline{v}}{\partial \overline{t}} = (-\overline{k}^2+1)\overline{v}.$$
Now we complete the proof of (31):
$$\begin{aligned}
&\frac{d}{d\overline{t}}\int_{\overline{\gamma}(\cdot,\overline{t})} \overline{\rho}\, d\overline{s} = \frac{d}{d\overline{t}}\int_{\overline{\gamma}(\cdot,\overline{t})} e^{-\frac{1}{2}\langle \overline{\gamma}(p,\overline{t}), \overline{\gamma}(p,\overline{t})\rangle}\,\overline{v}\, dp \\
&= \int_{\overline{\gamma}(\cdot,\overline{t})} \left[\overline{\rho}(-\overline{k}^2+1)\overline{v} + \overline{\rho}(-1)\left\langle \overline{\gamma}, \frac{\partial \overline{\gamma}}{\partial \overline{t}}\right\rangle \overline{v}\right] dp \\
&= \int_{\overline{\gamma}(\cdot,\overline{t})} \left[\overline{\rho}\left(-\left|\frac{\partial^2 \overline{\gamma}}{\partial \overline{s}^2}\right|^2 + 1\right) - \overline{\rho}\left\langle \overline{\gamma}, \frac{\partial^2 \overline{\gamma}}{\partial \overline{s}^2} + \overline{\gamma}\right\rangle\right] d\overline{s} \\
= &\int_{\overline{\gamma}(\cdot,\overline{t})} \left[-\overline{\rho}\left(\left|\frac{\partial^2 \overline{\gamma}}{\partial \overline{s}^2} + \overline{\gamma}\right|^2 - 2\left\langle \overline{\gamma}, \frac{\partial^2 \overline{\gamma}}{\partial \overline{s}^2}\right\rangle - |\overline{\gamma}|^2\right) + \overline{\rho} - \overline{\rho}\left(\left\langle \overline{\gamma}, \frac{\partial^2 \overline{\gamma}}{\partial \overline{s}^2}\right\rangle + |\overline{\gamma}|^2\right)\right] d\overline{s} \\
&= \int_{\overline{\gamma}(\cdot,\overline{t})} \left[-\overline{\rho}\left|\frac{\partial^2 \overline{\gamma}}{\partial \overline{s}^2} + \overline{\gamma}\right|^2 + \overline{\rho} + \overline{\rho}\left\langle \overline{\gamma}, \frac{\partial^2 \overline{\gamma}}{\partial \overline{s}^2}\right\rangle\right] d\overline{s}
\end{aligned}$$

$$
\begin{aligned}
&= \int_{\overline{\gamma}(\cdot,\overline{t})} \left[-\overline{\rho} \left| \frac{\partial^2 \overline{\gamma}}{\partial \overline{s}^2} + \overline{\gamma} \right|^2 + \overline{\rho} \frac{\partial}{\partial \overline{s}} \left\langle \overline{\gamma}, \frac{\partial \overline{\gamma}}{\partial \overline{s}} \right\rangle \right] d\overline{s} \\
&= \int_{\overline{\gamma}(\cdot,\overline{t})} -\overline{\rho} \left| \frac{\partial^2 \overline{\gamma}}{\partial \overline{s}^2} + \overline{\gamma} \right|^2 d\overline{s} - \int_{\overline{\gamma}(\cdot,\overline{t})} \frac{\partial \overline{\rho}}{\partial \overline{s}} \left\langle \overline{\gamma}, \frac{\partial \overline{\gamma}}{\partial \overline{s}} \right\rangle d\overline{s} \\
&= \int_{\overline{\gamma}(\cdot,\overline{t})} -\overline{\rho} \left(\left\langle \frac{\partial^2 \overline{\gamma}}{\partial \overline{s}^2} \frac{\partial^2 \overline{\gamma}}{\partial \overline{s}^2} \right\rangle + 2 \left\langle \frac{\partial^2 \overline{\gamma}}{\partial \overline{s}^2}, \overline{\gamma} \right\rangle + \langle \overline{\gamma}, \overline{\gamma} \rangle \right) d\overline{s} - \int_{\overline{\gamma}(\cdot,\overline{t})} \overline{\rho}(-1) \left\langle \overline{\gamma}, \frac{\partial \overline{\gamma}}{\partial \overline{s}} \right\rangle^2 d\overline{s} \\
&= \int_{\overline{\gamma}(\cdot,\overline{t})} \left[-\overline{\rho} \left(\left\langle \frac{\partial^2 \overline{\gamma}}{\partial \overline{s}^2} \frac{\partial^2 \overline{\gamma}}{\partial \overline{s}^2} \right\rangle + 2 \left\langle \frac{\partial^2 \overline{\gamma}}{\partial \overline{s}^2}, \overline{\gamma}^{\perp} \right\rangle + \left\langle \overline{\gamma}^{\perp}, \overline{\gamma}^{\perp} \right\rangle + \left\langle \overline{\gamma}^{\top}, \overline{\gamma}^{\top} \right\rangle \right) \right. \\
&\qquad \left. + \overline{\rho} \left\langle \overline{\gamma}, \frac{\partial \overline{\gamma}}{\partial \overline{s}} \right\rangle^2 \right] d\overline{s} \\
&= \int_{\overline{\gamma}(\cdot,\overline{t})} \left[-\overline{\rho} \left| \frac{\partial^2 \overline{\gamma}}{\partial \overline{s}^2} + \overline{\gamma}^{\perp} \right|^2 - \overline{\rho} \left\langle \overline{\gamma}, \frac{\partial \overline{\gamma}}{\partial \overline{s}} \right\rangle^2 + \overline{\rho} \left\langle \overline{\gamma}, \frac{\partial \overline{\gamma}}{\partial \overline{s}} \right\rangle^2 \right] d\overline{s} \\
&= \int_{\overline{\gamma}(\cdot,\overline{t})} -\overline{\rho} \left| \frac{\partial^2 \overline{\gamma}}{\partial \overline{s}^2} + \overline{\gamma}^{\perp} \right|^2 d\overline{s}. \qquad \square
\end{aligned}
$$

We will use Theorem 4.2(2) to study the behavior of $\overline{\gamma}(\cdot,\overline{t})$ as $\overline{t} \to \infty$. First notice that $\overline{\gamma}(\cdot,\overline{t})$ cannot disappear at infinity. Let $\overrightarrow{0} \in \mathbb{R}^2$ be the blow-up point. Then we have

$$|\gamma(p,t)| \leq \int_t^{\omega} |k| d\tau \leq C\sqrt{\omega - t},$$

and so,

$$|\overline{\gamma}(p,\overline{t})| \leq C.$$

Now, integrating the monotonicity formula in time gives the following lemma.

Lemma 4.2.

$$\int_{t_0}^{\infty} \int_{\overline{\gamma}(\cdot,\overline{t})} \overline{\rho} \left| \frac{\partial^2 \overline{\gamma}}{\partial \overline{s}^2} + \overline{\gamma}^{\perp} \right|^2 d\overline{s} d\overline{t} < \infty.$$

Corollary 4.1. $\forall \epsilon > 0, \quad \exists T < \infty$ *such that*

$$\int_{T}^{\infty} \int_{\overline{\gamma}(\cdot,\overline{t})} \overline{\rho} \left| \frac{\partial^2 \overline{\gamma}}{\partial \overline{s}^2} + \overline{\gamma}^{\perp} \right|^2 d\overline{s} d\overline{t} < \epsilon.$$

From the corollary above we get

$$\left| \frac{\partial^2 \overline{\gamma}}{\partial \overline{s}^2} + \overline{\gamma}^{\perp} \right| = 0.$$

That is,

$$\overline{k} = < \overline{\gamma}, -N > .$$

Hence the limit is an asymptotically self-similar homothetically shrinking solution. These are classified by Abresch and Langer [1], and the only embedded one is the circle. Hence if the forming singularity is type-I, then the curve converges to a round point in the C^∞ sense.

4.1.2. Type-II Singularities

We will now assume a type-II singularity is forming at time ω. Our model for this type of behavior is the formation of a cusp.

We will use the re-scaling from previous section. By [2], the limit solution γ_∞ exists for all time and the curvature k satisfies $0 < k \leq 1$, and $k = 1$ at the origin at $t = 0$.

Remark. The solution does not cross itself or else a loop would pinch and the curvature would not be bounded for all time in the future. The curve must turn at least π or else the curve would not be ancient (that is, it could not exist since $t = -\infty$). Thus the curve must turn exactly π and is embedded. So $\int k\, ds = \pi$, imply that the curvature goes to zero at the end of the curve. It is not hard to show that all of the derivatives of k must also decay to zero near the ends.

Therefore, by [17], the limit is a translating soliton. It is then necessarily the graph $y = f(x,t)$ of a function where

$$\frac{\partial y}{\partial t} = \frac{\partial}{\partial x} \tan^{-1}\left(\frac{\partial y}{\partial x}\right) = 1,$$

which is solved to give the grim reaper

$$y = t + \ln(\sec x).$$

In the grim reaper, a horizontal line segment has length $L < \pi$, while if it is high enough, it encloses an arbitrarily large area A_1, while there is still an arbitrarily large area A_2 on the other side if we go out far enough. If the grim reaper is to be the limit, then the original curve comes arbitrarily close to it after translating, rotating, and dilating; all of which do not affect the constant $\overline{G}$. But then we must have $\overline{G} \to 0$, which is impossible.

Thus we have proved the following main theorem.

Main Theorem. *Let γ be a closed embedded curve evolving by curvature flow on a smooth compact Riemannian surface. If a singularity develops in finite* *time, then the curve converges to a round point in the C^∞ sense.*

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