Evolution of Curves by Curvature Flow

by
Murugiah Muraleetharan

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Dissertation Director,
David L. Johnson (Chair)

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Committee Members

Huai-Dong Cao

Donald M. Davis

Ethan Berkove

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Contents

Abstract

Recently, new proofs of Grayson's theorem [Gra87] for curvature flow of embedded curves in the plane have been given by Hamilton [Ham95b] and Huisken [Hui98]. Hamilton proved this using monotonicity of isoperimetric estimates, and Huisken proved it by obtaining a lower bound for the quotient of the extrinsic distance in the plane by the intrinsic distance along the curve.

In this thesis, we will extend Grayson's theorem [Gra89] for the curvature flow of embedded curves in a compact Riemannian surface, by showing, if a singularity develops in finite time, then the curve converges to a round point in the C^∞ sense. We give two different proofs; one using Hamilton's isoperimetric estimates technique and the other one using Huisken's distance comparison technique.

KEYWORDS: Curvature flow, singularities

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Chapter 1

Introduction

Traditionally, differential geometry has been the study of curved spaces or shapes in which, for the most part, time did not play any role. In the last few decades, on the other hand, geometers have made great strides in understanding shapes that evolve in time. There are many processes by which a curve or surface can evolve, but among them, one is arguably the most natural: the mean curvature flow, where the evolution of the curve is in the direction of the principal normal, with magnitude given by the curvature. This flow is, in a sense, the gradient flow for the arclength functional. Thus, roughly speaking, the curve evolves so as to reduce its arclength as rapidly as possible.

In the past two decades, Richard Hamilton's Ricci flow has received attention as having a profound influence on geometric evolution equations and as a possible approach to studying Thurston's Geometrization Conjecture.

In 2002, Grisha Perelman claimed to have proved Thurston's geometrization conjecture using Hamilton's Ricci flow program — experts have been checking the details of the proof.

Partial differential equations play a major role in modern differential geometry. In particular, parabolic equations (geometric heat flows) have been successfully employed to improve geometric quantities. The flows chosen are typically the steepest decent, or gradient flows for the geometric energies considered.

We now state our main results in this thesis: Let γ be a closed embedded curve evolving under the curvature flow in a compact surface M. If a singularity develops in finite time, then the curve shrinks to a point. So when t is close enough to the blow-up time ω, we may assume that the curve is contained in a small neighborhood of the collapsing point on the surface. Using a local conformal diffeomorphism $\phi : U(\subseteq M) \to U' \subseteq \mathbb{R}^2$ between compact neighborhoods, we get a corresponding flow in the plane which satisfies the following equation:

$$\frac{\partial \gamma'}{\partial t} = \left(\frac{k'}{J^2} - \frac{\nabla_N J}{J^2} \right) N' \tag{1.1}$$

where $\gamma'(p,t) = \phi(\gamma(p,t))$, k' is the curvature of γ' in U', and N' is the unit normal vector.

For a smooth embedded closed curve γ in $\mathbb{R}^2$, consider any curve Γ which divides the region enclosed by γ into two pieces with areas A_1 and A_2, where $A_1 + A_2 = A$ is the area enclosed by γ. Let L be the length of Γ. We define

$$G(\gamma, \Gamma) = L^2 \left(\frac{1}{A_1} + \frac{1}{A_2} \right), \quad \text{and} \quad \overline{G}(\gamma) = \inf_{\Gamma} G(\gamma, \Gamma).$$

First, we prove the following lemma in section 3.1.

Lemma A. *If $\gamma'(\cdot, t)$ is evolving by the parabolic flow (1.1), and t_0 is close enough to the blow-up time $\omega < \infty$, then there is some $\varepsilon > 0$ such that $\overline{G}(\gamma'(\cdot, t)) > \epsilon$ for all $t \in [t_0, \omega)$.*

We define the extrinsic and intrinsic distance functions

$$d, l : \Gamma \times \Gamma \times [0, T] \to \mathbb{R}$$

by

$$d(p, q, t) = |\gamma(p, t) - \gamma(q, t)|_{\mathbb{R}^2} \quad \text{and} \quad l(p, q, t) = \int_p^q ds_t = s_t(q) - s_t(p)$$

where Γ is either S^1 or an interval.

In section 4.1 we prove the following lemma.

Lemma B. *Let $\gamma : I \times [0, T] \to \mathbb{R}^2$ be a smooth embedded solution of the flow (1.1), where I is an interval such that l is smoothly defined on $I \times I$. Suppose $\frac{d}{l}$ attains a local minimum at (p_0, q_0) in the interior of $I \times I$ at time $t_0 \in [0, T]$. Then*

$$\frac{d}{dt} \left(\frac{d}{l} \right) (p_0, q_0, t_0) \geq 0,$$

with equality if and only if γ is a straight line.

We now define the smooth function

$$\psi : S^1 \times S^1 \times [0,T] \to \mathbb{R}$$

by

$$\psi : (p,q,t) := \frac{L(t)}{\pi} \sin\left(\frac{l(p,q,t)\pi}{L(t)}\right).$$

We next prove the following lemma in section 4.2.

Lemma C. *Let $\gamma : S^1 \times [0,T] \to \mathbb{R}^2$ be a smooth embedded solution of the flow (1.1). Suppose $\frac{d}{\psi}$ attains a local minimum $(\frac{d}{\psi})(p_0, q_0, t_0) < 1$ at some point $(p_0, q_0) \in S^1 \times S^1$ at time $t_0 \in [0,T]$. Then*

$$\frac{d}{dt}\left(\frac{d}{\psi}\right)(p_0, q_0, t_0) \geq 0,$$

with equality if and only if $\frac{d}{\psi} \equiv 1$ or $\gamma(S^1, \cdot)$ is a circle.

Main Theorem. *Let γ be a closed embedded curve evolving by curvature flow on a smooth compact Riemannian surface. If a singularity develops in finite time, then the curve converges to a round point in the C^∞ sense.*

We will prove our main theorem, first, using lemma A in chapter 3, and then using lemmas B and C in chapter 4.

1.1 Evolving Closed Curves in a Plane

The simplest evolution problem is the evolution of curves in the plane by curvature flow.

Let γ_0 be a given smooth embedded convex closed plane curve, and let $\gamma : S^1 \times [0, \omega) \to \mathbb{R}^2$ be a one-parameter smooth family of embedded curves satisfying $\gamma(\cdot, 0) = \gamma_0$. If k is the curvature and $\boldsymbol{\nu}$ is the outward unit normal, then we say that γ evolves by the *curvature flow* (or curve shortening flow) if

$$\frac{\partial \gamma}{\partial t}(p,t) = -k(p,t)\boldsymbol{\nu}(p,t), \quad (p,t) \in S^1 \times [0, \omega). \tag{1.2}$$

The curvature vector $\mathbf{k}$ is defined by $\mathbf{k} = -k\boldsymbol{\nu}$. If we let $s = s_t$ be the arclength on $\gamma_t = \gamma(\cdot, t)$, then $\mathbf{k} = (\partial^2/\partial s^2)\gamma$, and the equation (1.2) can be written in the form $(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial s_t^2})\gamma = 0$, making the quasilinear parabolic nature of the equation apparent. So the evolution makes the curve smoother since such smoothing is a general feature of solutions to parabolic equations [Eva98]. Thus, for example, even if the initial curve is only C^2, as it starts moving, it immediately becomes C^∞ and indeed real analytic. Because the equation of motion is nonlinear, the general theory of parabolic equations does not preclude later singularities. And indeed, as we shall see, any curve must eventually become singular under the curvature flow. The existence, regularity, and long term behavior of solutions to this system have been extensively studied.

For any evolution of a curve and for its arclength L, we have

$$\frac{dL}{dt} = \int_{\gamma_t} \left\langle \frac{\partial \gamma}{\partial t}, k\boldsymbol{\nu} \right\rangle ds.$$

So for the curvature flow, $\frac{dL}{dt} = -\int_{\gamma_t} k^2 ds$. Indeed, this flow is, in a sense, the gradient flow for the arclength functional. Thus, roughly speaking, the curve evolves so as to reduce its arclength as rapidly as possible. This explains the name "curve shortening flow" though many other flows also reduce arclength.

For any evolution of a curve and the area A it encloses, we have

$$\frac{dA}{dt} = \int_{\gamma_t} \left\langle \frac{\partial \gamma}{\partial t}, \boldsymbol{\nu} \right\rangle ds.$$

So for the curvature flow, $\frac{dA}{dt} = -2\pi$ until the curve becomes singular. Thus a singularity must develop within $\frac{A(0)}{2\pi}$. Another important property is that the flow is collision-free. This collision avoidance is a special case of the maximum principle for parabolic differential equations. The maximum principle also implies in the same way that any initial embedded curve must remain embedded. The first deep theorem about curvature flow was proved by Gage and Hamilton [GH86].

Theorem 1.1.1. *[GH86] Under the curvature flow, a convex curve in a plane remains convex and shrinks to a point. Furthermore, it becomes asymptotically circular: If the evolving curve is dilated to keep the enclosed area con-*

stant, then the re-scaled curve converges smoothly to a circle, i.e., curves shrink to round points.

Consider a one-parameter family of embedded curves $\gamma : S^1 \times [0, \omega) \to \mathbb{R}^2$ evolving by the curvature flow. That is,

$$\frac{\partial \gamma}{\partial t}(p,t) = k(p,t)N(p,t), \quad (p,t) \in S^1 \times [0,\omega),$$

where N is its unit inward normal vector. Arclength is given by

$$s(p,t) = \int_0^p \left| \frac{\partial \gamma}{\partial q}(q,t) \right| dq.$$

Differentiating,

$$\frac{\partial s}{\partial p}(p,t) = \left| \frac{\partial \gamma}{\partial p}(p,t) \right| = v(p,t)$$

$$\Rightarrow \frac{\partial}{\partial s} = \frac{1}{v}\frac{\partial}{\partial p}, \quad \text{and} \quad ds = v dp.$$

We now state some standard results for the curvature flow.

Lemma 1.1.1. *For the curvature flow:*

1. *The speed v evolves according to $\frac{\partial v}{\partial t} = -k^2 v$.*

2. $\frac{\partial}{\partial t}\frac{\partial}{\partial s} = \frac{\partial}{\partial s}\frac{\partial}{\partial t} + k^2 \frac{\partial}{\partial s}$.

3. *The arclength L of the curve evolves according to $\frac{dL}{dt} = -\int_{\gamma_t} k^2 ds$.*

4. *The curvature k of the curve evolves according to $\frac{\partial k}{\partial t} = \frac{\partial^2 k}{\partial s^2} + k^3$.*

5. The enclosed area A of the curve evolves according to $\frac{dA}{dt} = -2\pi$.

Lemma 1.1.2 (Long Time Existence)**.** *Let γ be a solution of the curvature flow on the time interval $[0, \alpha)$. If k is bounded on $[0, \alpha)$, then $\exists\, \epsilon > 0$ such that $\gamma(\cdot, t)$ is a smooth solution on the time interval $[0, \alpha + \epsilon)$.*

For the body of this thesis, we will assume that the solution to the flow exists on the maximal time interval $[0, \omega)$.

1.2 Convex Curves in the Plane

We now consider convex curves in the plane. In this case, the curvature flow problem is equivalent to an initial value problem for a certain non-linear parabolic differential equation.

For a convex curve we use the angle θ of the tangent line as a parameter, and get the following non-linear parabolic differential equation:

$$\frac{\partial k}{\partial t} = k^2 \frac{\partial^2 k}{\partial \theta^2} + k^3 \tag{1.3}$$

The curvature flow problem is equivalent to this initial value PDE problem [GH86].

Gage [Gag83] showed that for the curvature flow, the isoperimetric ratio $\frac{L^2}{A}$ decreases, so that if $A \to 0$, then $L \to 0$ and the curve shrinks to a point. Gage [Gag84] then showed that the isoperimetric ratio $\frac{L^2}{A}$ decreases to its optimum value 4π as the enclosed area approaches 0, and, as a consequence, the ratio $\frac{r_{out}}{r_{in}}$ of the circumscribed radius to inscribed radius goes to 1. That is, the curve shrinks to a round point in the "C^0" sense if the enclosed area approaches 0.

In [GH86], Gage and Hamilton obtained the a priori estimates needed to show long term existence of the solution to the equation (1.3), showing that convex curves shrink to round points in the "C^∞" sense: First, they proved that when the enclosed area A is bounded away from 0, the curvature k is uniformly bounded, and if k is bounded, then $\frac{\partial k}{\partial \theta}$ and all the higher derivatives of k are also bounded. So as long as A is bounded away from 0, one gets bounds on k and all of its derivatives. Using the evolution equation, one can also bound the time derivatives. So, suppose the solution exists on the time interval $[0, T)$, and $\lim_{t \to T} A(t) > 0$. Then k has a limit as $t \to T$, and one can extend the solution past T. Thus the solution continues until the area goes to 0.

Finally, by re-scaling the curve so that the curve encloses constant area π and using a priori estimates, they showed that the higher derivatives of the curvature κ for the normalized curve converge to 0 and, therefore, the curve converges to a unit circle in the C^∞ sense.

1.3 Non-Convex Curves in the Plane

For higher dimensions, Huisken [Hui84] proved that under mean curvature flow, a convex hypersurface in $\mathbb{R}^{n+1}$ contracts smoothly to a single point in finite time, and becomes spherical at the end of the contraction. This result is not generally true for nonconvex embedded hypersurfaces. A barbell with a long, thin handle develops a singularity in the middle in short time. But under curvature flow for curves, Grayson [Gra87] showed that the assumption of the convexity of the initial curve can be removed, and he proved that the result holds for arbitrary smooth embedded closed initial curves by showing embedded curves become convex without developing singularities.

Theorem 1.3.1. *[Gra87] Under the curvature flow, an embedded curve in a plane becomes convex and thus eventually shrinks to a round point.*

The main ingredients of the proof contain the following: The solution remains smooth and embedded as long as its curvature remains bounded, the proof of the nonexistence of corners, which says that, if the curvature blows up anywhere, then it does so along an arc which has a total curvature of at least π, and the δ-whisker lemma, which says that, under certain conditions, the curve cannot get too close to itself. In the end, everything has been ruled out, except the case of the curve becoming convex before it becomes singular.

Recently, new direct proofs of Grayson's theorem [Gra87] for curvature flow of embedded curves in planes have been given by Hamilton [Ham95b] and Huisken [Hui98]. Hamilton proved this using monotonicity of isoperimetric estimates, and Huisken proved it by obtaining a lower bound for the quotient of the extrinsic distance in the plane to the intrinsic distance along the curve.

It is also important to study the way in which solutions can become singular, and methods for continuing solutions through a singularity:

A natural classification of singularities arises from the blow-up rate of the curvature into type I and type II singularities [Alt91, Ang91a]. In a type I singularity, $|k|_{max}\sqrt{\omega - t}$ is bounded, where ω is the blow-up time, the singularity looks asymptotically like a contracting self-similar solution. In a type II singularity, $|k|_{max}\sqrt{\omega - t}$ is unbounded, there is a sequence of points and times $\{(p_n, t_n)\}$ on which the curve blows up such that a rescaling of the solution along this sequence converges to the Grim Reaper ($y = \ln \sec x$).

If the initial curve has self-intersections, then small loops may contract in short time, causing the curvature to become unbounded. That is, the corresponding solution can become singular without shrinking to a point. A typical example is the flow of a limaçon of Pascal ($r = 1 + a\cos\theta$, $a > 1$). The small loop of the limaçon of Pascal contracts and develops a cusp when the large loop still exists. In this case, the curve converges to some singular limit curve which is piecewise smooth, with a finite number of singularities.

Therefore to continue solutions through a singularity, the allowable set of initial curves should be expanded so that it contains limit curves, and there would be a solution for (1.2) which has a singular limit curve as initial data, in some weak sense.

In a series of important papers, Angenent [Ang90, Ang91b, Ang91a] showed that locally Lipschitz or even worse initial data can be used as initial curves, using more specialized theory of parabolic partial differential equations. Having dealt with the initial value problem, he then showed that a limit curve always exists, and that it is a locally Lipschitz curve with finite total absolute curvature. He also proved that the singular curves are nice enough that, with some possible trimming, they may be used as initial data for the curvature flow, and the number of self-intersections of an evolving curve can not increase with time. So a generalized solution of (1.2) that becomes singular at a discrete set of times, either exists forever, or shrinks to a point in finite time.

Chapter 2

Evolving Closed Curves in a Surface

2.1 Introduction

Grayson [Gra89] and Gage [Gag90], generalized the study of curvature flow of closed curves in the plane to that in surfaces. The curvature flow problem is to analyze the long term behavior of smooth curves immersed in a surface (M^2, g) with Riemannian metric g. The curvature flow is a gradient flow for the length functional on the space of immersed curves in the surface M^2. Therefore, one can try to use curvature flow to prove existence of closed geodesics by variational methods. Although evolution by curvature is a natural way to shorten the curves, it leads to number of complex problems. Can the curve do this at all: is there short-term existence? If so, for how long do the solutions exist? What do the limiting solutions look like?

2.2 Some Basic Results of the Evolution

Let (M, g) be a smooth compact oriented 2-dimensional Riemannian manifold with bounded scalar curvature. Let $\gamma_0 : S^1 \to M$ be a smooth embedded curve in M and let $\gamma : S^1 \times [0, \omega) \to M$ be a one parameter smooth family of embedded curves satisfying $\gamma(\cdot, 0) = \gamma_0$. If γ evolves by curvature flow, then

$$\frac{\partial \gamma}{\partial t}(p,t) = k(p,t)N(p,t), \quad (p,t) \in S^1 \times [0, \omega), \tag{2.1}$$

where k is the geodesic curvature of γ and N is its unit normal.

Arclength is given by

$$s(p,t) = \int_0^p \left| \frac{\partial \gamma}{\partial q}(q,t) \right| dq.$$

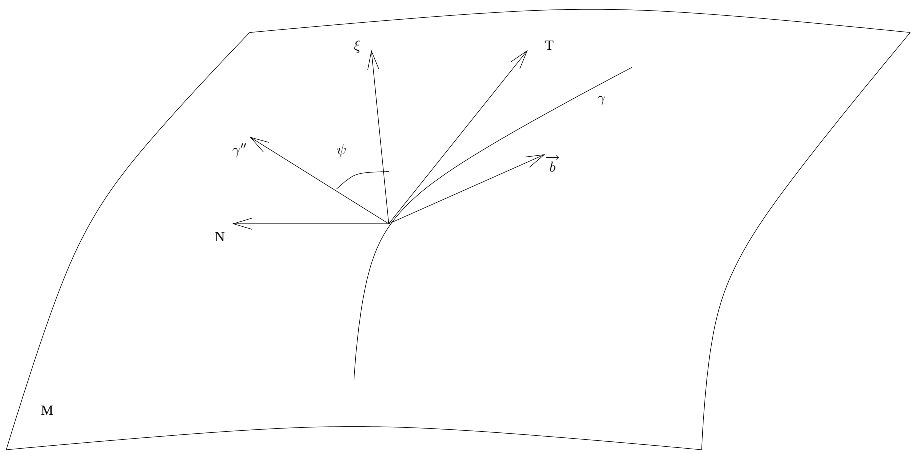
Differentiating,

$$\frac{\partial s}{\partial p}(p,t) = \left| \frac{\partial \gamma}{\partial p}(p,t) \right| = v(p,t)$$

$$\Rightarrow \frac{\partial}{\partial s} = \frac{1}{v}\frac{\partial}{\partial p}, \quad \text{and} \quad ds = v dp.$$

First we recall the curvature of a curve on a surface.

The curvature of a curve on a surface



Let $T = \frac{\partial \gamma}{\partial s}$, and ξ be the standard unit normal to the surface, and $N = \xi \times T$. That is, N is chosen such that (T, N) agrees with the orientation of the surface. Then T, N, and ξ are mutually perpendicular unit vectors and the Frenet formula is given by

$$\frac{dT}{ds} = k_n \xi + k_g N$$

$$\frac{dN}{ds} = -k_g T + \tau_g \xi$$

$$\frac{d\xi}{ds} = -k_n T - \tau_g N$$

where $k_n = \left\langle \frac{\partial^2 \gamma}{\partial s^2}, \xi \right\rangle$ is the normal curvature of the curve, $k_g = \left\langle \frac{\partial^2 \gamma}{\partial s^2}, N \right\rangle$ is the geodesic curvature of the curve, and τ_g is the geodesic torsion, given by $\tau_g = \tau - \frac{d\psi}{ds}$ where τ is the torsion, and ψ is the angle between ξ and $\frac{\partial^2 \gamma}{\partial s^2}$.

If we use k instead k_g for the geodesic curvature, then from Frenet formula, we have

$$\nabla_s T = kN \quad \text{and} \quad \nabla_s N = -kT.$$

Now we recall some standard results for the evolution [Gra89].

Lemma 2.2.1. *For the curvature flow:*

1. *The speed v evolves according to $\frac{\partial v}{\partial t} = -k^2 v$.*
2. $[\frac{\partial}{\partial t}, \frac{\partial}{\partial s}] = k^2 \frac{\partial}{\partial s}$.
3. $\nabla_t T = \frac{\partial k}{\partial s} N$ *and* $\nabla_t N = -\frac{\partial k}{\partial s} T$.
4. *The arclength L of the curve evolves according to $\frac{dL}{dt} = -\int_{\gamma_t} k^2 ds$.*
5. $\nabla_t \nabla_s = \nabla_s \nabla_t + k^2 \nabla_s - kR(T, N)$.
6. *The curvature k of the curve evolves according to $\frac{\partial k}{\partial t} = \frac{\partial^2 k}{\partial s^2} + k^3 + K\, k$, where $K = \langle R(N,T)T, N) \rangle$ is the Gaussian curvature of M restricted to $\gamma(\cdot, t)$.*
7. *The enclosed area A of the curve evolves according to $\frac{dA}{dt} = -2\pi$.*

Now we state the main theorem in [Gra89].

Theorem 2.2.1. *[Gra89] A closed embedded curve moving on a smooth compact Riemannian surface by curvature flow must either collapse to a point in finite time or else converge to a simple closed geodesic as $t \to \infty$.*

Grayson first showed that the solution remains smooth and embedded as long as its curvature remains bounded. He then proved that if a singularity develops in finite time, then the curvature remains bounded until the entire curve shrinks to a point. Finally, he proved that if the length of the curve does not converge to zero, then its curvature must converge to zero in the C^∞ norm and that the curve approaches a geodesic in the C^∞ sense.

In this thesis, we will extend Grayson's theorem [Gra89] for curvature flow of embedded curves in a compact Riemannian surface, by showing that if the curve shrinks to a point, then it shrinks to a round point in the C^∞ sense. We give two different proofs; one using Hamilton's isoperimetric estimates technique, and the other one using Huisken's distance comparison technique.

In order to apply the above techniques, we first need to transform the curvature flow in surfaces to a corresponding flow in the plane. We will see in the next section that the corresponding flow is no longer the curvature flow. It is a more general flow. In a series of papers, Angenent [Ang90, Ang91b, Ang91a] developed a more general theory of parabolic equations for curves on surfaces. We now summarize some of the important results of Angenent that we will need.

2.3 Parabolic Equations for Curves on Surfaces

Consider a closed curve evolving by an arbitrary uniformly parabolic equation,

$$\frac{\partial \gamma}{\partial t} = V(T, k)N, \tag{2.2}$$

on a smooth oriented 2-dimensional Riemannian manifold M, and denote its unit tangent bundle by $S^1(M) = \{\xi \in T(M) : g(\xi, \xi) = 1\}$. Then the normal velocity is

$$v^{\perp}(p, t) = V(T, k)(p, t) \equiv V(T_{\gamma(p,t)}, k_{\gamma(p,t)}),$$

for some function $V : S^1(M) \times \mathbb{R} \to \mathbb{R}$ which satisfies:

(V_1) $V(T, k)$ is $C^{2,1}$,

(V_2) $\lambda^{-1} \leq \dfrac{\partial V}{\partial k} \leq \lambda$,

(V_3) $|V(T, 0)| \leq \mu \quad$ for all $T \in S^1(M)$,

(V_4) $|\nabla^h V| + |k\nabla^v V| \leq \nu(1 + k^2)$,

(V_5) $V(-T, -k) = -V(T, k)$,

for positive constants λ, μ, and ν.

The tangent bundle to $S^1(M)$ splits into the Whitney sum of the bundle of horizontal vectors and bundle of vertical vectors. $\nabla^v V$ and $\nabla^h V$ denote the vertical and horizontal components of $\nabla(V)$ (holding the second argument of V fixed).

These assumptions on V are necessary to make the set of allowable initial curves as large as possible, and necessary for the short-time existence of the solutions. The way in which maximal classical solutions can become singular (limit curves) is based on these assumptions on V and the initial curves.

Examples:

1. $V(T, k) = k$, the curvature flow problem, and

2. $V(T, k) = \left(\frac{k}{J^2} - \frac{\nabla_N J}{J^2}\right)$, where $J(x, y)$ is a smooth bounded function that is also bounded away from 0. We will see that the curvature flow in a surface corresponds to the flow with this normal velocity in a plane.

First, we state some basic results similar to lemma 2.2.1.

Lemma 2.3.1. *For the flow 2.1:*

1. *The speed v evolves according to $\frac{\partial v}{\partial t} = -kVv$.*

2. $[\frac{\partial}{\partial t}, \frac{\partial}{\partial s}] = kV\frac{\partial}{\partial s}$.

3. $\nabla_t T = \frac{\partial V}{\partial s} N$, *and* $\nabla_t N = -\frac{\partial V}{\partial s} T$.

4. *The arclength L of the curve evolves according to* $\frac{dL}{dt} = -\int_{\gamma_t} kV\,ds$.

5. $\nabla_t \nabla_s = \nabla_s \nabla_t + kV\nabla_s - VR(T, N)$.

6. *The curvature k of the curve evolves according to* $\frac{\partial k}{\partial t} = \frac{\partial^2 V}{\partial s^2} + k^2 V + KV$, *where* $K = \langle R(N, T)T, N)\rangle$ *is the Gaussian curvature of M restricted to* $\gamma(\cdot, t)$.

7. *The enclosed area A of the curve evolves according to* $\frac{dA}{dt} = -\int_{\gamma_t} V\,ds$.

We now state the next four results from [Ang90] and [Ang91b].

Theorem 2.3.1. *If V satisfies $(V_1) - (V_4)$, then for any locally Lipschitz initial curve γ_0 there is a family of curves $\gamma : S^1 \times [0, \omega) \to M$ which satisfies (2.2), and has the initial position $\gamma(\cdot, 0) = \gamma_0$.*

Theorem 2.3.2. *Let V satisfy $(V_1) - (V_5)$. Then the initial value problem (2.2) has a short time solution for any initial curve which is locally C^1 and regular.*

Having dealt with the initial value problem, we now state the way in which a maximal solution of (2.2) can become singular.

Theorem 2.3.3. *Let V satisfy $(V_1) - (V_5)$, and let $\gamma : S^1 \times [0, \omega) \to M$ be a maximal classical solution of (2.2). Then the limit curve γ^* of the $\gamma(\cdot, t)$ is a piecewise C^1 curve, which is $C^{2,\alpha}$ away from its singular points.*

Theorem 2.3.4. *Let V satisfy $(V_1) - (V_5)$, and let $\gamma : S^1 \times [0, \omega) \to M$ be a maximal classical solution of (2.2) which becomes singular in finite time. Then the limit curve γ^* of the $\gamma(\cdot, t)$ either has fewer self-intersections than any of the $\gamma(\cdot, t)$'s, or else the total absolute curvature of the limit curve drops by at least π.*

Oaks [Oak94], improved Theorem 2.3.4 by showing that the latter case never occurs. So if the initial curve is embedded, and the singularity develops in finite time, then the curve shrinks to a point. So when t is close enough to the blow-up time ω, we may assume that the curve is contained in a small neighborhood of the collapsing point on the surface.

Now from the following theorem, it is enough to work locally in $\mathbb{R}^2$.

Theorem 2.3.5. *[Oak94] Let $\phi : U(\subseteq M) \to U' \subseteq \mathbb{R}^2$ be a conformal diffeomorphism between compact neighborhoods. If $V : S^1(M) \times \mathbb{R} \to \mathbb{R}$ satisfies $(V_1) - (V_5)$, then there is a function $V' : S^1(U') \times \mathbb{R} \to \mathbb{R}$ which satisfies $(V_1) - (V_5)$ such that whenever $\gamma(p, t)$ is a curve in U evolving by (2.2), $\gamma'(p, t) = \phi(\gamma(p, t))$ satisfies $\frac{\partial \gamma'}{\partial t} = V'(T', k')N'$, where T' and N' are the unit tangent and normal vectors, and k' is the curvature of γ' in U'.*

Moreover, $V(T, k) = J(p)V'(T', k')$ and $ds = J(p)ds'$, where $J(p) > 0$ is smooth, bounded, and bounded away from 0.

The metric in U can be written as

$$g = J^2(x,y)(dx^2 + dy^2),$$

where the coordinates in U are obtained by ϕ^{-1}. Because U' is compact, $J(x,y)$ is both bounded and bounded away from 0.

Let $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ be the coordinate vector fields on U, and let $X = \frac{1}{J}\frac{\partial}{\partial x}$, and $Y = \frac{1}{J}\frac{\partial}{\partial y}$. Then X and Y are unit vectors. Since ϕ is conformal, $\phi_*(N) = \frac{1}{J}N'$. So γ' evolves by the equation:

$$\frac{\partial \gamma'}{\partial t} = (\frac{1}{J}V)N'.$$

Therefore, $V' = \frac{1}{J}V$.

We next show that $k' = kJ + \nabla_N J$. First, we need the following lemma.

Lemma 2.3.2.

$$\nabla_X X = -\frac{\nabla_Y J}{J}Y \qquad \nabla_X Y = \frac{\nabla_Y J}{J}X$$
$$\nabla_Y X = \frac{\nabla_X J}{J}Y \qquad \nabla_Y Y = -\frac{\nabla_X J}{J}X.$$

Proof

Since $0 = [\frac{\partial}{\partial x}, \frac{\partial}{\partial y}] = [JX, JY]$, we have $\nabla_{JX}JY = \nabla_{JY}JX$. Therefore, $J\nabla_X Y + (\nabla_X J)Y = J\nabla_Y X + (\nabla_Y J)X$. Since $\nabla_X Y \perp Y$ and $\nabla_Y X \perp X$, we get $\nabla_X Y = \frac{\nabla_Y J}{J}X$ and $\nabla_Y X = \frac{\nabla_X J}{J}Y$. The other two formulas follow from differentiating $\langle X, Y\rangle = 0$ with respect to X and Y. $\square$

Let θ be the angle T makes with X in U. Then

$$T = \cos\theta X + \sin\theta Y, \quad N = -\sin\theta X + \cos\theta Y.$$

Thus,

$$\begin{aligned}
\nabla_T X &= \cos\theta\,\nabla_X X + \sin\theta\,\nabla_Y X \\
&= -\cos\theta\,\frac{\nabla_Y J}{J} Y + \sin\theta\,\frac{\nabla_X J}{J} Y \\
&= \left(-\frac{\nabla_Y J}{J}\cos\theta + \frac{\nabla_X J}{J}\sin\theta\right) Y,
\end{aligned}$$

and

$$\nabla_T Y = \left(\frac{\nabla_Y J}{J}\cos\theta - \frac{\nabla_X J}{J}\sin\theta\right) X.$$

We have $\nabla_T\theta = \frac{1}{J}k'$. Then

$$\begin{aligned}
&kN = \gamma'' = \nabla_T T = \nabla_T(\cos\theta\,X + \sin\theta\,Y) \\
&\quad = -\sin\theta(\nabla_T\theta)X + \cos\theta(\nabla_T X) + \cos\theta(\nabla_T\theta)Y + \sin\theta(\nabla_T Y) \\
&\quad = -\sin\theta(\frac{k'}{J})X + \cos\theta\left(-\frac{\nabla_Y J}{J}\cos\theta + \frac{\nabla_X J}{J}\sin\theta\right) Y + \cos\theta(\frac{k'}{J})Y \\
&\qquad + \sin\theta\left(\frac{\nabla_Y J}{J}\cos\theta - \frac{\nabla_X J}{J}\sin\theta\right) X \\
&\quad = \left(\frac{k'}{J} + \frac{\nabla_X J}{J}\sin\theta - \frac{\nabla_Y J}{J}\cos\theta\right)(-\sin\theta\,X + \cos\theta\,Y) \\
&\quad = \left(\frac{k'}{J} + \frac{\nabla_X J}{J}\sin\theta - \frac{\nabla_Y J}{J}\cos\theta\right) N,
\end{aligned}$$

and thus,

$$\begin{aligned} k &= \left(\frac{k'}{J} + \frac{\nabla_X J}{J} \sin\theta - \frac{\nabla_Y J}{J} \cos\theta \right) \\ &= \frac{k'}{J} - \frac{1}{J}(-\sin\theta \nabla_X J + \cos\theta \nabla_Y J) \\ &= \frac{k'}{J} - \frac{1}{J} \nabla_N J. \end{aligned}$$

That is,

$$k' = kJ + \nabla_N J. \tag{2.3}$$

J is bounded away from 0 and both J and $\nabla_N J$ are bounded. So $\lim_{t\to\omega} |k(p,t)|$ is unbounded if and only if $\lim_{t\to\omega} |k'(p,t)|$ is also unbounded.

When $V = k$, i.e., for the curvature flow in a surface M, we have $V' = \frac{1}{J}V = \frac{k}{J} = \frac{k'}{J^2} - \frac{\nabla_N J}{J^2}$. So the curvature flow in a surface corresponds to the following flow in $\mathbb{R}^2$:

$$\frac{\partial \gamma'}{\partial t} = \left(\frac{k'}{J^2} - \frac{\nabla_N J}{J^2} \right) N'. \tag{2.4}$$

Chapter 3

Formation of the Singularity

From the previous chapter, we know that when a closed curve evolves under the curvature flow in a surface, the solution remains smooth and embedded as long as its curvature remains bounded. If a singularity develops in finite time, then the curve shrinks to a point. So when t is close enough to the blow-up time ω, we may assume that the curve is contained in a small neighborhood of the collapsing point on the surface. Now by theorem 2.3.5, using a local conformal diffeomorphism $\phi : U(\subseteq M) \to U' \subseteq \mathbb{R}^2$ between compact neighborhoods, we get a corresponding flow in the plane which satisfies the following equation:

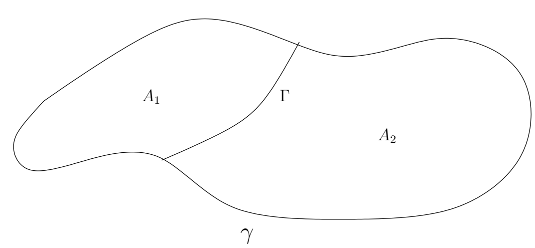
$$\frac{\partial \gamma'}{\partial t} = \left(\frac{k'}{J^2} - \frac{\nabla_N J}{J^2} \right) N', \tag{3.1}$$

where $\gamma'(p,t) = \phi(\gamma(p,t))$, k' is the curvature of γ' in U', and N' is the unit normal vector.

Hamilton [Ham95b] showed that a certain isoperimetric ratio $\overline{G}(\gamma(\cdot,t))$ improves under the curvature flow in the plane when $\overline{G}(\gamma(\cdot,t)) \leq \pi$. In the next section, we will prove Lemma A by showing that the isoperimetric ratio $\overline{G}(\gamma'(\cdot,t))$ improves under the parabolic flow (3.1).

3.1 Monotonicity of an Isoperimetric Ratio

For a smooth embedded closed curve γ in $\mathbb{R}^2$, consider any curve Γ which divides the region enclosed by γ into two pieces with areas A_1 and A_2, where $A_1 + A_2 = A$ is the area enclosed by γ. Let L be the length of Γ. Define the ratio



$$G(\gamma, \Gamma) = L^2 \left(\frac{1}{A_1} + \frac{1}{A_2} \right), \tag{3.2}$$

and let

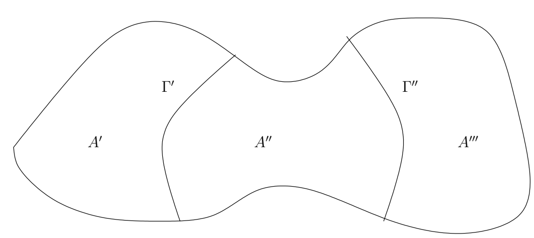
$$\overline{G}(\gamma) = \inf_{\Gamma} G(\gamma, \Gamma) \tag{3.3}$$

be the least possible value of $G(\gamma, \Gamma)$ for all curve segments Γ. Hamilton [Ham95b] takes the infimum over all possible straight lines for (3.3). He also defines another isoperimetric ratio, and for that ratio he takes the infimum over all possible curves. We will use the following theorem of Hamilton and its proof.

Lemma 3.1.1. *[Ham95b] The minimum $\overline{G}(\gamma)$ is attained by a single smooth curve Γ_0 of constant curvature perpendicular to γ.*

Proof

For any division of area $A = A_1 + A_2$, there will be a shortest curve (or collection of curves) effecting this division, and the curve (or each component curve) will have constant curvature and be perpendicular to the boundary γ. It is among such curves that we see one Γ of minimum length $L(\Gamma)$, and since this set is compact we can surely find one. Now we show the best Γ will have only one component.



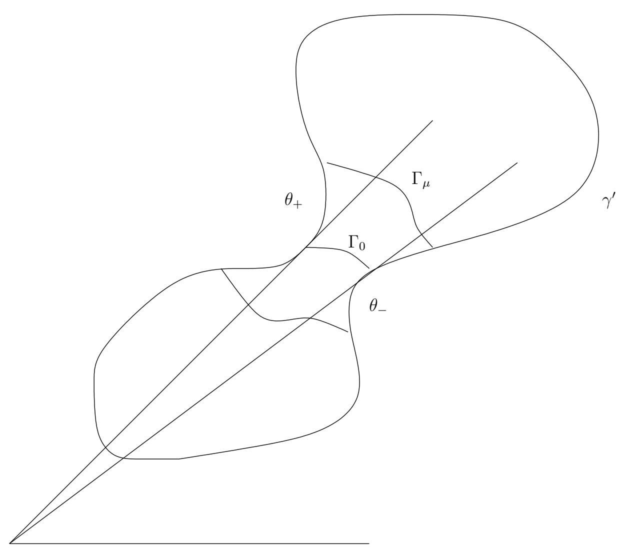
Suppose for example, that Γ has two components Γ' and Γ'' of lengths L' and L'', dividing A into regions of area $A' + A'' + A''' = A$. Take $L = L' + L''$ and $A_1 = A' + A'''$ and $A_2 = A''$. Then we have

$$(L'+L'')^2\left(\frac{1}{A''}+\frac{1}{A'+A'''}\right) \geq \min\left\{L'^2\left(\frac{1}{A'}+\frac{1}{A''+A'''}\right),\quad L''^2\left(\frac{1}{A'''}+\frac{1}{A'+A''}\right)\right\}.$$

So using only one part of the curve for the division gives smaller ratio. Hence the best Γ will have only one component. $\square$

We now show, in the rest of this section, that the isoperimetric ratio $\overline{G}(\gamma'(\cdot, t))$ improves under the parabolic flow (3.1).

Let's fix the time at $t = t_0$, and consider any one-parameter family of curves Γ_μ with parameter $\mu \in [-\mu_0, \mu_0]$, $\mu_0 > 0$. We will compute the first and second variation of the length $L(\Gamma_\mu)$ and the areas $A_1(\Gamma_\mu)$ and $A_2(\Gamma_\mu)$. We assume Γ_0 is our arc of constant curvature which gives the infimum for $G(\gamma'(\cdot, t_0), \Gamma_\mu)$ at $\mu = 0$.



In polar coordinates, Γ_μ is given by the graph of

$$r = r(\theta, \mu), \quad \theta \in [\theta_-(\mu), \theta_+(\mu)],$$

where θ_- is the portion of γ' near where it meets the bottom of Γ_0 and θ_+ is the portion of γ' near where it meets the top of Γ_0. So,

$$\Gamma_0 = \{(r_0, \theta) : r_0 = \frac{1}{K_0}, \theta \in [\theta_-(0), \theta_+(0)]\},$$

and we have

$$\left.\frac{\partial r}{\partial \theta}\right|_{\mu=0} = 0, \quad \text{and} \quad \left.\frac{\partial^2 r}{\partial \theta^2}\right|_{\mu=0} = 0. \tag{3.4}$$

Since Γ_0 is perpendicular to $\gamma'(\cdot, t_0)$ at $\mu = 0$, and we have

$$\left.\frac{\partial \theta_+}{\partial \mu}\right|_{\mu=0} = 0, \quad \text{and} \quad \left.\frac{\partial \theta_-}{\partial \mu}\right|_{\mu=0} = 0. \tag{3.5}$$

The curve γ' has curvatures k'_+ at $\theta_+(0)$ and k'_- at $\theta_-(0)$ which can be computed as the curvatures of the graphs of

$$\theta_+(\mu), \quad r_+(\mu) = r(\theta_+(\mu), \mu)$$

and

$$\theta_-(\mu), \quad r_-(\mu) = r(\theta_-(\mu), \mu).$$

The curvature of a parameterized curve $P(\mu) = (r\cos\theta, r\sin\theta)$ is given by

$$k = \frac{|P'(\mu) \times P''(\mu)|}{|P'(\mu)|^3}.$$

By using the above formula, we get

$$r_0 \left.\frac{d^2\theta_+}{d\mu^2}\right|_{\mu=0} = -k_+^{'} \left(\left.\frac{\partial r_+}{\partial \mu}\right|_{\mu=0} \right)^2, \tag{3.6}$$

and

$$r_0 \frac{d^2\theta_-}{d\mu^2}\bigg|_{\mu=0} = k_+^{'} \left(\frac{\partial r_-}{\partial \mu}\bigg|_{\mu=0} \right)^2. \tag{3.7}$$

For the variation, let the velocity $v = \frac{\partial r}{\partial \mu}\big|_{\mu=0}$ and the acceleration $z = \frac{\partial^2 r}{\partial \mu^2}\big|_{\mu=0}$.

The arclength is given by

$$\begin{aligned} L(\mu) = L(\Gamma_\mu) &= \int_{\theta_-(\mu)}^{\theta_+(\mu)} \left| \frac{\partial P}{\partial \theta} \right| d\theta \\ &= \int_{\theta_-(\mu)}^{\theta_+(\mu)} \sqrt{r^2 + \left(\frac{\partial r}{\partial \theta} \right)^2}\, d\theta. \end{aligned}$$

Therefore,

$$\begin{aligned} \frac{dL}{d\mu} = &\int_{\theta_-(\mu)}^{\theta_+(\mu)} \frac{1}{2} \left(r^2 + \left(\frac{\partial r}{\partial \theta} \right)^2 \right)^{-1/2} \left(2r \frac{\partial r}{\partial \mu} + 2 \frac{\partial r}{\partial \theta} \frac{\partial^2 r}{\partial \theta \partial \mu} \right) d\theta \\ &+ \sqrt{r^2 + \left(\frac{\partial r}{\partial \theta} \right)^2}\Bigg|_{\theta_+(\mu)} \frac{\partial \theta_+}{\partial \mu} - \sqrt{r^2 + \left(\frac{\partial r}{\partial \theta} \right)^2}\Bigg|_{\theta_-(\mu)} \frac{\partial \theta_-}{\partial \mu}. \end{aligned}$$

Thus,

$$\frac{dL}{d\mu}\bigg|_{\mu=0} = \int_{\theta_-(0)}^{\theta_+(0)} \frac{\partial r}{\partial \mu}\bigg|_{\mu=0} d\theta.$$

That is,

$$\frac{dL}{d\mu}\bigg|_{\mu=0} = \int_{\theta_-(0)}^{\theta_+(0)} v\, d\theta. \tag{3.8}$$

Now consider the second variation of L.

$$\frac{d^2L}{d\mu^2} = \int_{\theta_-(\mu)}^{\theta_+(\mu)} \Bigg[-\frac{1}{2}\left(r^2 + \left(\frac{\partial r}{\partial \theta}\right)^2\right)^{-3/2} 2\left(r\frac{\partial r}{\partial \mu} + \frac{\partial r}{\partial \theta}\frac{\partial^2 r}{\partial\theta\partial\mu}\right)^2$$

$$+ \left(r^2 + \left(\frac{\partial r}{\partial \theta}\right)^2\right)^{-1/2} \left(\left(\frac{\partial r}{\partial \mu}\right)^2 + r\frac{\partial^2 r}{\partial \mu^2} + \left(\frac{\partial^2 r}{\partial\theta\partial\mu}\right)^2 + \frac{\partial r}{\partial \theta}\frac{\partial^3 r}{\partial\theta\partial\mu^2}\right)\Bigg] d\theta$$

$$+ \sqrt{r^2 + \left(\frac{\partial r}{\partial \theta}\right)^2}\,\Bigg|_{\theta_+(\mu)} \frac{\partial^2 \theta_+}{\partial \mu^2} + \frac{d}{d\mu}\left(\sqrt{r^2 + \left(\frac{\partial r}{\partial \theta}\right)^2}\,\Bigg|_{\theta_+(\mu)}\right)\frac{\partial \theta_+}{\partial \mu}$$

$$+ \sqrt{r^2 + \left(\frac{\partial r}{\partial \theta}\right)^2}\,\Bigg|_{\theta_-(\mu)} \frac{\partial^2 \theta_-}{\partial \mu^2} + \frac{d}{d\mu}\left(\sqrt{r^2 + \left(\frac{\partial r}{\partial \theta}\right)^2}\,\Bigg|_{\theta_-(\mu)}\right)\frac{\partial \theta_-}{\partial \mu}.$$

Thus,

$$\frac{d^2L}{d\mu^2}\bigg|_{\mu=0} = \int_{\theta_-(0)}^{\theta_+(0)} z\, d\theta + K_0 \int_{\theta_-(0)}^{\theta_+(0)} \left(\frac{dv}{d\theta}\right)^2 d\theta - (k_+^{'} v_+^2 + k_-^{'} v_-^2). \qquad (3.9)$$

Now we compute the first and second variation of the areas A_1 and A_2. Since $A_1(\mu) + A_2(\mu) = A$, we have $\frac{dA_1}{d\mu} = -\frac{dA_2}{d\mu}$ and $\frac{d^2A_1}{d\mu^2} = -\frac{d^2A_2}{d\mu^2}$. We have

$$\text{Area} \quad A = \int\int r dr d\theta = \int\int r\frac{\partial r}{\partial \mu} d\mu d\theta.$$

If $A_1(\mu)$ denotes the area on the origin side of Γ_μ, then

$$A_1(\mu) - A_1(0) = \int_0^\mu \int_{\theta_-(\tau)}^{\theta_+(\tau)} r\frac{\partial r}{\partial \tau} d\theta d\tau.$$

Therefore,

$$\frac{dA_1}{d\mu} = \int_{\theta_-(\mu)}^{\theta_+(\mu)} r\frac{\partial r}{\partial \mu}\, d\theta \quad \Rightarrow \quad \frac{dA_1}{d\mu}\bigg|_{\mu=0} = \frac{1}{K_0}\int_{\theta_-(0)}^{\theta_+(0)} v\, d\theta,$$

and we have

$$\left.\frac{dA_1}{d\mu}\right|_{\mu=0} = -\left.\frac{dA_2}{d\mu}\right|_{\mu=0} = \frac{1}{K_0}\int_{\theta_-(0)}^{\theta_+(0)} v\,d\theta. \tag{3.10}$$

Now,

$$\frac{d^2A_1}{d\mu^2} = \int_{\theta_-(\mu)}^{\theta_+(\mu)} r\frac{\partial^2 r}{\partial\mu^2} + \left(\frac{\partial r}{\partial\mu}\right)^2 d\theta + r\left.\frac{\partial r}{\partial\mu}\right|_{\theta_+(\mu)}\frac{\partial\theta_+}{\partial\mu} - r\left.\frac{\partial r}{\partial\mu}\right|_{\theta_-(\mu)}\frac{\partial\theta_-}{\partial\mu}.$$

Thus,

$$\left.\frac{d^2A_1}{d\mu^2}\right|_{\mu=0} = -\left.\frac{d^2A_2}{d\mu^2}\right|_{\mu=0} = \frac{1}{K_0}\int_{\theta_-(0)}^{\theta_+(0)} z\,d\theta + \int_{\theta_-(0)}^{\theta_+(0)} v^2\,d\theta. \tag{3.11}$$

Having found the first and second variations of L, A_1 and A_2, we can now write down the condition that $G = L^2\left(\frac{1}{A_1}+\frac{1}{A_2}\right)$ attains its minimum at Γ_0. As usual, this says that $\left.\frac{dG}{d\mu}\right|_{\mu=0} = 0$ and $\left.\frac{d^2G}{d\mu^2}\right|_{\mu=0} \geq 0$. It is easier to express this inequality in terms of logarithms; thus

$$G = L^2\left(\frac{1}{A_1}+\frac{1}{A_2}\right) = L^2\left(\frac{A}{A_1A_2}\right) \Rightarrow \ln G = 2\ln L + \ln A - \ln A_1 - \ln A_2.$$

Since

$$0 = \left.\frac{d}{d\mu}\right|_{\mu=0}\ln G = \frac{2}{L}\left.\frac{dL}{d\mu}\right|_{\mu=0} + \frac{1}{A}\left.\frac{dA}{d\mu}\right|_{\mu=0} - \frac{1}{A_1}\left.\frac{dA_1}{d\mu}\right|_{\mu=0} - \frac{1}{A_2}\left.\frac{dA_2}{d\mu}\right|_{\mu=0},$$

we have

$$\frac{2}{L}\int_{\theta_-(0)}^{\theta_+(0)} v\,d\theta - \frac{1}{A_1}\frac{1}{K_0}\int_{\theta_-(0)}^{\theta_+(0)} v\,d\theta + \frac{1}{A_2}\frac{1}{K_0}\int_{\theta_-(0)}^{\theta_+(0)} v\,d\theta = 0.$$

Therefore,

$$\frac{2K_0}{L} = \frac{1}{A_1} - \frac{1}{A_2}. \tag{3.12}$$

Next, we have

$$
\begin{aligned}
0 \leq \frac{d^2}{d\mu^2}\bigg|_{\mu=0} \ln G = & \frac{d}{d\mu}\bigg|_{\mu=0} \left(\frac{2}{L}\frac{dL}{d\mu} - \frac{1}{A_1}\frac{dA_1}{d\mu} - \frac{1}{A_2}\frac{dA_2}{d\mu} \right) \\
= & \frac{d}{d\mu}\bigg|_{\mu=0} \left(\frac{2}{L}\frac{dL}{d\mu} - \frac{dA_1}{d\mu}\left(\frac{1}{A_1} - \frac{1}{A_2} \right) \right) \\
= & \left[-\frac{2}{L^2}\left(\frac{dL}{d\mu}\right)^2 + \frac{2}{L}\frac{d^2L}{d\mu^2} - \frac{d^2A_1}{d\mu^2}\left(\frac{1}{A_1} - \frac{1}{A_2}\right) + \frac{dA_1}{d\mu}\left(\frac{1}{A_1^2}\frac{dA_1}{d\mu} - \frac{1}{A_2^2}\frac{dA_2}{d\mu} \right) \right]_{\mu=0} \\
= & -\frac{2}{L^2}\left(\int_{\theta_-(0)}^{\theta_+(0)} v\, d\theta \right)^2 + \frac{2}{L}\left[\int_{\theta_-(0)}^{\theta_+(0)} z\, d\theta + K_0 \int_{\theta_-(0)}^{\theta_+(0)} \left(\frac{dv}{d\theta}\right)^2 d\theta - (k'_+ v_+^2 + k'_- v_-^2) \right] \\
& - \left(\frac{1}{A_1} - \frac{1}{A_2} \right)\left[\frac{1}{K_0}\int_{\theta_-(0)}^{\theta_+(0)} z\, d\theta + \int_{\theta_-(0)}^{\theta_+(0)} v^2\, d\theta \right] + \left(\frac{1}{A_1^2} + \frac{1}{A_2^2} \right)\left[\frac{1}{K_0}\int_{\theta_-(0)}^{\theta_+(0)} v\, d\theta \right]^2 \\
= & \left[-\frac{2}{L^2} + \frac{1}{K_0^2}\left(\frac{1}{A_1^2} + \frac{1}{A_2^2} \right) \right]\left(\int_{\theta_-(0)}^{\theta_+(0)} v\, d\theta \right)^2 + \left[\frac{2}{L} - \frac{1}{K_0}\left(\frac{1}{A_1} - \frac{1}{A_2} \right) \right]\int_{\theta_-(0)}^{\theta_+(0)} z\, d\theta \\
& + \frac{2K_0}{L}\int_{\theta_-(0)}^{\theta_+(0)} \left(\frac{dv}{d\theta}\right)^2 d\theta - \frac{2}{L}(k'_+ v_+^2 + k'_- v_-^2) - \left(\frac{1}{A_1} - \frac{1}{A_2} \right)\int_{\theta_-(0)}^{\theta_+(0)} v^2\, d\theta.
\end{aligned}
$$

So, by using (3.12), we get

$$
\begin{aligned}
\frac{2}{L}(k'_+ v_+^2 + k'_- v_-^2) \leq & \frac{1}{2K_0^2}\left(\frac{1}{A_1} + \frac{1}{A_2} \right)^2 \left(\int_{\theta_-(0)}^{\theta_+(0)} v\, d\theta \right)^2 \\
& + \frac{2K_0}{L}\int_{\theta_-(0)}^{\theta_+(0)} \left(\left(\frac{dv}{d\theta}\right)^2 - v^2 \right) d\theta. \quad (3.13)
\end{aligned}
$$

The curvature flow in a surface corresponds to the following flow in the plane $\mathbb{R}^2$ (see (3.1)),

$$
\frac{\partial \gamma'}{\partial t} = \left(\frac{k'}{J^2} - \frac{\nabla_N J}{J^2} \right) N' = V'N'.
$$

We will now use (3.12) and (3.13) to show that $\frac{d}{dt}\big|_{t=t_0} \ln G > 0$. First we need to compute the evolution of L, A, A_1 and A_2 at time t_0. The evolution of the length L is the sum of the normal velocity of $\gamma'(\cdot, t)$ at the two ends of Γ_0, so that

$$\left.\frac{dL}{dt}\right|_{t=t_0} = -\left(V'_+ + V'_-\right).$$

The evolution of the areas are given by:

$$\left.\frac{dA}{dt}\right|_{t=t_0} = -\int_{\gamma'(\cdot,t_0)} V'\,ds, \quad \left.\frac{dA_1}{dt}\right|_{t=t_0} = -\int_{\gamma'_1(\cdot,t_0)} V'\,ds, \quad \left.\frac{dA_2}{dt}\right|_{t=t_0} = -\int_{\gamma'_2(\cdot,t_0)} V'\,ds.$$

Since

$$G = L^2\left(\frac{1}{A_1} + \frac{1}{A_2}\right) = L^2\frac{A}{A_1A_2},$$

we have

$$\ln G = 2\ln L + \ln A - \ln A_1 - \ln A_2,$$

and

$$\left.\frac{d}{dt}\right|_{t=t_0} \ln G = \frac{2}{L}\left.\frac{dL}{dt}\right|_{t=t_0} + \frac{1}{A}\left.\frac{dA}{dt}\right|_{t=t_0} - \frac{1}{A_1}\left.\frac{dA_1}{dt}\right|_{t=t_0} - \frac{1}{A_2}\left.\frac{dA_2}{dt}\right|_{t=t_0}.$$

Thus,

$$\begin{aligned}\left.\frac{d}{dt}\right|_{t=t_0} \ln G = &-\frac{2}{L}\left(\frac{k'_+}{J_+^2} + \frac{k'_-}{J_-^2}\right) + \frac{2}{L}\left(\left.\frac{\nabla_N J}{J^2}\right|_+ + \left.\frac{\nabla_N J}{J^2}\right|_-\right) \\ &- \frac{1}{A}\int_{\gamma'(\cdot,t_0)} V'\,ds + \frac{1}{A_1}\int_{\gamma'_1(\cdot,t_0)} V'\,ds + \frac{1}{A_2}\int_{\gamma'_2(\cdot,t_0)} V'\,ds. \quad (3.14)\end{aligned}$$

If we choose the variation such that

$$v(r_0, \theta) = \frac{1}{J(r_0\cos\theta, r_0\sin\theta)} \quad \theta \in [\theta_-(0), \theta_+(0)],$$

then we could use the result from (3.13) in (3.14). Consider the RHS of (3.13). First,

$$\int_{\theta_-(0)}^{\theta_+(0)} v\, d\theta \leq \max_{\theta_-(0)\leq\theta\leq\theta_+(0)} \left(\frac{1}{J}\right)(\theta_+ - \theta_-).$$

Using $(\theta_+ - \theta_-) = LK_0$, we get that the first term of the RHS of (3.13) is bounded by

$$\frac{1}{2K_0^2}\left(\frac{1}{A_1}+\frac{1}{A_2}\right)^2 \left(\int_{\theta_-(0)}^{\theta_+(0)} v\, d\theta\right)^2 \leq \frac{C_1 L^2}{2}\left(\frac{1}{A_1}+\frac{1}{A_2}\right)^2, \qquad (3.15)$$

where

$$C_1 = \max_{\theta_-\leq\theta\leq\theta_+} \left(\frac{1}{J^2}\right). \qquad (3.16)$$

Now considering the second term in the RHS of (3.13), we have

$$\begin{aligned}\frac{dv}{d\theta} &= -\frac{1}{J^2}[J_x(-r_0\sin\theta) + J_y(r_0\cos\theta)]\\ &\leq \frac{r_0}{J^2}\sqrt{J_x^2+J_y^2},\end{aligned}$$

so we get the bound

$$\frac{2K_0}{L}\int_{\theta_-(0)}^{\theta_+(0)} \left(\frac{dv}{d\theta}\right)^2 d\theta \leq \frac{2K_0}{L}\frac{C_2}{K_0^2}(LK_0) = 2C_2, \qquad (3.17)$$

where

$$C_2 = \max_{\theta_-\leq\theta\leq\theta_+} \left(\frac{J_x^2+J_y^2}{J^4}\right). \qquad (3.18)$$

Now using (3.12) we get the bound for the third term in the RHS of (3.13),

$$\frac{2K_0}{L}\int_{\theta_-(0)}^{\theta_+(0)} v^2\, d\theta \geq \frac{2K_0}{L}C_3(LK_0) = 2C_3K_0^2 = \frac{C_3L^2}{2}\left(\frac{1}{A_1}-\frac{1}{A_2}\right)^2, \quad (3.19)$$

where

$$C_3 = \min_{\theta_- \leq \theta \leq \theta_+} \left(\frac{1}{J^2} \right). \tag{3.20}$$

So now (3.13), (3.15), (3.17), and (3.19) give

$$\frac{2}{L} \left(\frac{k'_+}{J_+^2} + \frac{k'_-}{J_-^2} \right) \leq \frac{C_1 L^2}{2} \left(\frac{1}{A_1} + \frac{1}{A_2} \right)^2 + 2C_2 - \frac{C_3 L^2}{2} \left(\frac{1}{A_1} - \frac{1}{A_2} \right)^2. \tag{3.21}$$

So we have a bound on the first term in the RHS of (3.14). We now bound the last three terms in the RHS of (3.14). First,

$$\begin{aligned} \int_{\gamma'} V' \, ds &= \int_{\gamma'} \left(\frac{k'}{J^2} - \frac{\nabla_N J}{J^2} \right) ds \\ &\geq C_4 \int_{\gamma'} k' \, ds - C_5 \int_{\gamma'} ds, \end{aligned}$$

where

$$C_4 = \min_{U'} \left(\frac{1}{J^2} \right), \tag{3.22}$$

and

$$C_5 = \max_{U'} \left(\frac{\nabla_N J}{J^2} \right). \tag{3.23}$$

Therefore,

$$
\begin{aligned}
&-\frac{1}{A}\int_{\gamma'(\cdot,t_0)} V'\,ds+\frac{1}{A_1}\int_{\gamma_1'(\cdot,t_0)} V'\,ds+\frac{1}{A_2}\int_{\gamma_2'(\cdot,t_0)} V'\,ds\\
&=\left(\frac{1}{A_1}-\frac{1}{A}\right)\int_{\gamma_1'}\left(\frac{k'}{J^2}-\frac{\nabla_N J}{J^2}\right)ds+\left(\frac{1}{A_2}-\frac{1}{A}\right)\int_{\gamma_2'}\left(\frac{k'}{J^2}-\frac{\nabla_N J}{J^2}\right)ds\\
&\geq\left(\frac{1}{A_1}-\frac{1}{A}\right)\left[C_4\int_{\gamma_1'}k'\,ds-C_5\int_{\gamma_1'}ds\right]+\left(\frac{1}{A_2}-\frac{1}{A}\right)\left[C_4\int_{\gamma_2'}k'\,ds-C_5\int_{\gamma_2'}ds\right]\\
&=\left(\frac{1}{A_1}-\frac{1}{A}\right)[C_4(\pi-(\theta_+-\theta_-))-C_5L(\gamma_1')]\\
&\qquad+\left(\frac{1}{A_2}-\frac{1}{A}\right)[C_4(\pi+(\theta_+-\theta_-))-C_5L(\gamma_2')]\\
&=C_4\pi\left(\frac{A_2}{A_1A}+\frac{A_1}{A_2A}\right)+C_4(\theta_+-\theta_-)\left(-\frac{A_2}{A_1A}+\frac{A_1}{A_2A}\right)\\
&\qquad-C_5\left[\left(\frac{1}{A_1}-\frac{1}{A}\right)L(\gamma_1')+\left(\frac{1}{A_2}-\frac{1}{A}\right)L(\gamma_2')\right]\\
&=\frac{C_4\pi(A_1^2+A_2^2)}{A_1A_2(A_1+A_2)}+\frac{C_4L^2}{2}\left(\frac{1}{A_1}-\frac{1}{A_2}\right)\frac{(A_1^2-A_2^2)}{A_1A_2(A_1+A_2)}+C_5\left[\frac{L(\gamma')}{A}-\frac{L(\gamma_1')}{A_1}-\frac{L(\gamma_2')}{A_2}\right].
\end{aligned}
$$

Since

$$\frac{1}{2}(A_1+A_2)^2\leq(A_1^2+A_2^2)\leq(A_1+A_2)^2,$$

we have

$$\frac{(A_1+A_2)^2}{2A_1A_2(A_1+A_2)}\leq\frac{(A_1^2+A_2^2)}{A_1A_2(A_1+A_2)}\leq\frac{(A_1+A_2)^2}{A_1A_2(A_1+A_2)};$$

that is,

$$\frac{1}{2}\left(\frac{1}{A_1}+\frac{1}{A_2}\right)\leq\frac{(A_1^2+A_2^2)}{A_1A_2(A_1+A_2)}\leq\left(\frac{1}{A_1}+\frac{1}{A_2}\right).$$

Also,

$$\begin{aligned}
&\frac{L(\gamma')}{A} - \frac{L(\gamma_1')}{A_1} - \frac{L(\gamma_2')}{A_2} \\
&= \frac{L_1 + L_2}{A_1 + A_2} - \frac{L_1}{A_1} - \frac{L_2}{A_2} \\
&= -\frac{A_2^2 L_1 + A_1^2 L_2}{A_1 A_2 (A_1 + A_2)} \\
&\geq -\frac{L'(A_1^2 + A_2^2)}{A_1 A_2 (A_1 + A_2)} \\
&\geq -L' \left(\frac{1}{A_1} + \frac{1}{A_2} \right),
\end{aligned}$$

where $L' = \max(L_1, L_2)$.

So now we have bound on the last three terms in the RHS of (3.14):

$$\begin{aligned}
&-\frac{1}{A} \int_{\gamma'(\cdot,t_0)} V'\, ds + \frac{1}{A_1} \int_{\gamma_1'(\cdot,t_0)} V'\, ds + \frac{1}{A_2} \int_{\gamma_2'(\cdot,t_0)} V'\, ds \\
&\geq \frac{C_4 \pi}{2} \left(\frac{1}{A_1} + \frac{1}{A_2} \right) - \frac{C_4 L^2}{2} \left(\frac{1}{A_1} - \frac{1}{A_2} \right)^2 - C_5 L' \left(\frac{1}{A_1} + \frac{1}{A_2} \right). \quad (3.24)
\end{aligned}$$

Next we compute the second term in the RHS of (3.14). Let

$$\frac{\nabla_N J}{J^2}(\theta) = \frac{\nabla_N J}{J^2}(r_0 \cos\theta, r_0 \sin\theta, \theta).$$

Then

$$\begin{aligned}
\left. \frac{\nabla_N J}{J^2} \right|_{-} &= \frac{\nabla_N J}{J^2}(r_0 \cos\theta_-, r_0 \sin\theta_-, \theta_-) \\
&= \frac{\nabla_N J}{J^2}(\theta_-),
\end{aligned}$$

and

$$
\begin{aligned}
\left.\frac{\nabla_N J}{J^2}\right|_+ &= \frac{\nabla_N J}{J^2}(r_0\cos\theta_+, r_0\sin\theta_+, \theta_+ + \pi) \\
&= -\frac{\nabla_N J}{J^2}(r_0\cos\theta_+, r_0\sin\theta_+, \theta_+) \\
&= -\frac{\nabla_N J}{J^2}(\theta_+).
\end{aligned}
$$

By the mean value theorem,

$$
\left(\left.\frac{\nabla_N J}{J^2}\right|_+ + \left.\frac{\nabla_N J}{J^2}\right|_-\right) = \left(\frac{\nabla_N J}{J^2}\right)'(\theta_0)(\theta_- - \theta_+),
$$

for some $\theta_0 \in (\theta_-, \theta_+)$. Therefore

$$
\begin{aligned}
\frac{2}{L}\left(\left.\frac{\nabla_N J}{J^2}\right|_+ + \left.\frac{\nabla_N J}{J^2}\right|_-\right) &= -\frac{2}{L}\left(\frac{\nabla_N J}{J^2}\right)'(\theta_0)(LK_0) \\
&= -\left(\frac{\nabla_N J}{J^2}\right)'(\theta_0)L\left(\frac{1}{A_1} - \frac{1}{A_2}\right). \qquad (3.25)
\end{aligned}
$$

Thus (3.14), (3.21), (3.24), and (3.25) give

$$
\begin{aligned}
\left.\frac{d}{dt}\right|_{t=t_0} \ln G \geq& -\frac{C_1L^2}{2}\left(\frac{1}{A_1}+\frac{1}{A_2}\right)^2 - 2C_2 + \frac{C_3L^2}{2}\left(\frac{1}{A_1}-\frac{1}{A_2}\right)^2 \\
&+ \frac{C_4\pi}{2}\left(\frac{1}{A_1}+\frac{1}{A_2}\right) - \frac{C_4L^2}{2}\left(\frac{1}{A_1}-\frac{1}{A_2}\right)^2 - C_5L'\left(\frac{1}{A_1}+\frac{1}{A_2}\right) \\
&- \left(\frac{\nabla_N J}{J^2}\right)'(\theta_0)L\left(\frac{1}{A_1}-\frac{1}{A_2}\right) \\
=& \frac{1}{2}\left(\frac{1}{A_1}+\frac{1}{A_2}\right)\left(C_4\pi - C_1\overline{G} - 2C_5L' - 2\left(\frac{\nabla_N J}{J^2}\right)'(\theta_0)\,\frac{A_2-A_1}{A_1+A_2}\,L\right) \\
&+ \frac{L^2}{2}\left(\frac{1}{A_1}-\frac{1}{A_2}\right)^2(C_3 - C_4) - 2C_2. \qquad (3.26)
\end{aligned}
$$

If t_0 is close enough to the blow-up time ω, we can make C_4 and C_1 approach 1, and C_5 and C_2 approach 0. The term $\left(\frac{\nabla_N J}{J^2}\right)'(\theta_0)\,\frac{A_2-A_1}{A_1+A_2}$ is bounded. We also have $C_3 \geq C_4$. The lengths L' and L approach 0, and $\left(\frac{1}{A_1}+\frac{1}{A_2}\right)$ becomes larger. Hence when $\overline{G}$ gets smaller, $\frac{d}{dt}\Big|_{t=t_0} \ln G > 0$.

Thus we have proved the following main lemma.

Lemma A. *If $\gamma'(\cdot,t)$ is evolving by the parabolic flow (3.1), and t_0 is close enough to the blow-up time $\omega < \infty$, then there is some $\varepsilon > 0$ such that $\overline{G}(\gamma'(\cdot,t)) > \epsilon$ for all $t \in [t_0,\omega)$.*

In the next two sections we will study the formation of singularity by re-scaling the solutions, and then prove our main theorem using lemma A.

3.2 The Limit of the Re-Scaled Solutions

If the evolution equation has a smooth solution on a maximal time interval $0 \leq t < \omega < \infty$, then the supremum norm of the curvature must blow up as $t \to \omega$. We say that $Q \in \mathbb{R}^2$ is a *blow-up point or singularity* if there is $p \in S^1$ such that $\gamma(p,t) \to Q$ and $k(p,t)$ becomes unbounded as $t \to \omega$. We define $\{(p_n,t_n) \in S^1 \times [0,\omega)\}$ to be a *blow-up sequence* if $\lim_{n\to\infty} t_n = \omega$, $\lim_{n\to\infty} k(p_n,t_n) = \infty$, and

$$|k(p,t)| \leq |k(p_n,t_n)| \quad p \in S^1,\, t \in (0,t_n].$$

Let $M_t = \sup k^2(\cdot, t)$. Then we will use the following dilation-invariant categorization of singularity formation:

1. **Type-I** singularity if $\lim_{t \to \omega} M_t.(\omega - t)$ is bounded, and

2. **Type-II** singularity if $\lim_{t \to \omega} M_t.(\omega - t)$ is unbounded.

We next re-scale the solution along a blow-up sequence $\{(p_n, t_n)\}$: for every n we obtain a new solution γ_n, from γ by translating $t_n \mapsto 0$, and dilating the solution in space and time (scaling time as space squared) so that $k_n^2(p_n, 0) \mapsto 1$. First, we will give a precise definition: We have

$$\gamma : S^1 \times [0, \omega) \to \mathbb{R}^2.$$

We define the re-scaled solutions γ_n of γ along the blow-up sequence $\{(p_n, t_n)\}$ to be as follows:

$$\gamma_n : S^1 \times [-\lambda_n^2 t_n, \lambda_n^2(\omega - t_n)) \to \mathbb{R}^2$$

is given by

$$\gamma_n(\cdot, \bar{t}) = \lambda_n[\gamma(\cdot, t)] = \lambda_n[\gamma(\cdot, t_n + \lambda_n^{-2}\bar{t})]$$

where $\lambda_n = |k(p_n, t_n)|$ and $\bar{t} = \lambda_n^2(t - t_n)$. So, we have

$$t \in [0, \omega) \Leftrightarrow \bar{t} \in [-\lambda_n^2 t_n, \lambda_n^2(\omega - t_n)) = [a_n, \omega_n) \, \text{say}$$

That is,

$$\gamma_n : S^1 \times [a_n, \omega_n) \to \mathbb{R}^2$$

is given by

$$\gamma_n(\cdot,\bar{t}) = \lambda_n \gamma(\cdot,t).$$

Since $\lambda_n \to \infty$, we have $\lim_{n\to\infty} a_n = -\infty$ and

$$\lim_{n\to\infty} \omega_n = \begin{cases} \text{finite} & \text{if type I,} \\ +\infty & \text{if type II.} \end{cases}$$

The curvature of γ_n satisfies $|k_n(p,\bar{t})| \leq 1 \quad \text{for all} \quad \bar{t} \in [a_n, 0]$. We have

$$\frac{\partial \gamma_n}{\partial \bar{t}} = \lambda_n \frac{\partial \gamma}{\partial t}\frac{dt}{d\bar{t}} = \lambda_n \frac{\partial \gamma}{\partial t}(\lambda_n^{-2}).$$

So

$$\frac{\partial \gamma_n}{\partial \bar{t}} = \frac{1}{\lambda_n}\left(\frac{\partial \gamma}{\partial t}\right).$$

Since

$$\frac{\partial \gamma}{\partial t} = V(T,k)N = \left(\frac{k}{J^2} - \frac{\nabla_N J}{J^2}\right) N$$

(notice that we have dropped the prime), we have

$$\begin{aligned} \frac{\partial \gamma_n}{\partial \bar{t}} &= \frac{1}{\lambda_n} V(T, \lambda_n k_n) N \\ &= \left(\frac{k_n}{J^2} - \frac{1}{\lambda_n}\frac{\nabla_N J}{J^2}\right) N. \end{aligned}$$

A limit solution, if it exists, may be a family of noncompact curves. So think of our solutions as a family of $L(t)$ (length of $\gamma(\cdot,t)$) periodic curves,

$$\widetilde{\gamma_n} : \mathbb{R} \times [a_n, \omega_n) \to \mathbb{R}^2,$$

such that $\widetilde{\gamma_n}(0,\cdot) = \gamma_n(p_n,\cdot)$. We also parameterize the curves by arclength from the origin $0 \in \mathbb{R}$.

Now as in [Ang90], a uniform bound on the curvature implies bounds on the higher derivatives. Therefore, by the Ascoli-Arzela theorem one can extract a subsequence of $\widetilde{\gamma_n}(\cdot,t)$ which converges on compact sets of $\mathbb{R} \times (-\infty,\omega_\infty)$ to a smooth family of curves $\widetilde{\gamma_\infty}$.

The limit solution $\widetilde{\gamma_\infty}$ is either closed, or unbounded and complete. We will denote by γ_∞ one period, possibly infinite, of $\widetilde{\gamma_\infty}$, which satisfies

$$\frac{\partial \gamma_\infty}{\partial \bar{t}} = \frac{k_\infty}{J^2(Q)} N = k_\infty N,$$

where Q is the collapsing point of $\gamma(\cdot,t)$, and k_∞ is the curvature of $\gamma_\infty(\cdot,\bar{t})$. So $|k_\infty(p,\bar{t})| \leq 1$ for all $\bar{t} \leq 0$ with $|k_\infty(0,0)| = 1$, and hence the process of re-scaling does not allow the limit solution to be trivial, that is, a straight line.

Lemma 3.2.1. *[Alt91] For a closed embedded curve in the plane evolving by curvature flow we have*

$$\frac{d}{dt}\int_{\gamma(\cdot,t)} |k|\, ds = -2 \sum_{p:k(p,t)=0} \left| \frac{\partial k}{\partial s} \right|.$$

Theorem 3.2.1. γ_∞ *is a family of convex curves.*

Proof

On the limit solution, $\int_{\gamma_\infty(\cdot,t)} |k_\infty|\, ds$ is constant. We also have

$$\frac{d}{dt}\int_{\gamma_\infty(\cdot,t)} |k|\, ds = -2 \sum_{p:k_\infty(p,t)=0} \left|\frac{\partial k_\infty}{\partial s}\right|.$$

Hence,

$$\int_{-\infty}^{\omega_\infty} \sum_{p:k_\infty(p,t)=0} \left|\frac{\partial k_\infty}{\partial s}\right| dt = 0.$$

Therefore, any inflection points for the limit curve must be degenerate (i.e., $k_\infty = \frac{\partial k_\infty}{\partial s} = 0$). So [Ang91b] implies that if a solution has degenerate inflection points for any interval in time, then the solution must be a line. Since γ_∞ is not trivial, the family of curves must have no inflection points and therefore must all be convex. □

3.3 Limiting Shapes of Re-scaled Solutions along Blow-up Sequences

3.3.1 Type-I Singularities

In this section we assume $\{(p_n, t_n)\}$ is type-I blow-up sequence. We will prove that the re-scaled solutions γ_∞ on $[0, \omega_\infty)$ of the curvature flow converge to a solution which moves simply by homothety. It is convenient to drop the ∞ symbol in this section and consider the solution as $\gamma(p,t)$ on $[0,\omega)$.

The blow-up rate of the curvature still satisfies $M_t \leq \frac{C}{\omega - t}$. For curvature flow in a plane, we have the evolution equation for the curvature: $\frac{\partial k}{\partial t} = \frac{\partial^2 \gamma}{\partial s^2} + k^3$, and also $|k|_{\max}(t)$ has a lower bound $\frac{1}{\sqrt{2(\omega - t)}}$. Thus, the curvature of γ is uniformly pinched between two positive constants, and so all higher derivatives of the curvature are bounded as well.

Now we want to re-scale $\gamma(\cdot, t)$ near a singular point as $t \to \omega$, such that the re-scaled curve remains uniformly bounded. So we define the re-scaled solution $\overline{\gamma}$ of the solution γ on $[0, \omega)$ by

$$\overline{\gamma}(p, \bar{t}\,) = \frac{\gamma(p,t)}{\sqrt{2(\omega - t)}},$$

where

$$\bar{t} = -\frac{1}{2}\ln(\omega - t) \in \left[-\frac{1}{2}\ln\omega, +\infty\right) \equiv [t_0, +\infty).$$

That is,

$$\overline{\gamma} : S^1 \times [t_0, \infty) \to \mathbb{R}^2$$

and

$$\frac{d\bar{t}}{dt} = \frac{1}{2(\omega - t)} \Rightarrow \frac{\partial}{\partial \bar{t}} = 2(\omega - t)\frac{\partial}{\partial t}.$$

Arclength is given by

$$\overline{s}(p, \bar{t}) = \int_0^p \left| \frac{\partial \overline{\gamma}}{\partial q}(q, \bar{t}) \right| \, dq.$$

Differentiating,

$$\overline{v}(p,t) = \frac{\partial \overline{s}}{\partial p}(p, \bar{t}) = \left| \frac{\partial \overline{\gamma}}{\partial p}(p, \bar{t}) \right| = \frac{1}{\sqrt{2(\omega - t)}} \left| \frac{\partial \gamma}{\partial p}(p,t) \right|.$$

Thus,

$$\frac{\partial \overline{s}}{\partial p} = \frac{1}{\sqrt{2(\omega - t)}}\frac{\partial s}{\partial p} \Rightarrow \frac{\partial}{\partial \overline{s}} = \sqrt{2(\omega - t)}\frac{\partial}{\partial s}.$$

Hence we have the following operators:

$$\frac{\partial}{\partial \bar{t}} = 2(\omega - t)\frac{\partial}{\partial t}, \qquad \frac{\partial}{\partial \overline{s}} = \sqrt{2(\omega - t)}\frac{\partial}{\partial s}.$$

Therefore,

$$\begin{aligned}\frac{\partial \overline{\gamma}}{\partial \bar{t}} &= 2(\omega - t)\frac{\partial}{\partial t}\left(\frac{\gamma}{\sqrt{2(\omega - t)}}\right) \\ &= \sqrt{2(\omega - t)}\frac{\partial \gamma}{\partial t} + \frac{\gamma}{\sqrt{2(\omega - t)}} \\ &= \sqrt{2(\omega - t)}\frac{\partial^2 \gamma}{\partial s^2} + \overline{\gamma} \\ &= \frac{\partial}{\partial \overline{s}}\left(\frac{\partial \gamma}{\partial s}\right) + \overline{\gamma}.\end{aligned}$$

But,

$$\begin{aligned}\frac{\partial \overline{\gamma}}{\partial \overline{s}} &= \sqrt{2(\omega - t)}\frac{\partial}{\partial s}\left(\frac{\gamma}{\sqrt{2(\omega - t)}}\right) = \sqrt{2(\omega - t)}\frac{1}{v}\frac{\partial}{\partial p}\left(\frac{\gamma}{\sqrt{2(\omega - t)}}\right) \\ &= \sqrt{2(\omega - t)}\frac{1}{\sqrt{2(\omega - t)}}\frac{1}{v}\frac{\partial \gamma}{\partial p} = \frac{\partial \gamma}{\partial s},\end{aligned}$$

so the re-scaled solutions satisfy the equation

$$\frac{\partial \overline{\gamma}}{\partial \bar{t}} = \frac{\partial^2 \overline{\gamma}}{\partial \overline{s}^2} + \overline{\gamma}.$$

The curvature of the modified solution is

$$\overline{k}(p, \bar{t}) = \sqrt{2(\omega - t)}k(p, t).$$

Since we are assuming the forming singularity is type-I, the curvature $\overline{k}^2(\cdot, \bar{t})$ is uniformly bounded, and all higher derivatives of the curvature are bounded as well.

Monotonicity and self-similar solutions

Huisken [Hui90] proved a general monotonicity formula for hypersurfaces moving by mean curvature flow. Then he used the monotonicity result to show that singularities satisfying the growth rate estimate $M_t \leq \frac{C}{\omega - t}$ (type-I), are asymptotically self-similar. As in [Alt91], we apply the Huisken monotonicity formula for the curves evolving by curvature flow in a plane.

Let $\rho(x,t)$ be the backward heat kernel at $(\overrightarrow{0}, \omega)$, i.e.,

$$\rho(x,t) = \frac{1}{\sqrt{4\pi(\omega - t)}} \exp\left(\frac{-|x|^2}{4(\omega - t)}\right), \qquad t < \omega.$$

In the re-scaled setting we obtain a monotonicity formula if we define the modified kernel by

$$\overline{\rho}(x, \overline{t}) = e^{-\frac{1}{2}|x|^2}, \qquad x \in \mathbb{R}^2.$$

We now state the monotonicity formula:

Theorem 3.3.1. *[Hui90]*

1. *For γ, when $t \in [0, \omega)$, we have the formula*

$$\frac{d}{dt} \int_{\gamma(\cdot,t)} \rho(x,t)\, ds = - \int_{\gamma(\cdot,t)} \rho(x,t) \left| \frac{\partial^2 \gamma}{\partial s^2} + \frac{1}{2(\omega - t)} \gamma^{\perp} \right|^2 ds. \quad (3.27)$$

2. *For $\overline{\gamma}$, when $\overline{t} \in [t_0, \infty)$, we have the formula*

$$\frac{d}{d\overline{t}} \int_{\overline{\gamma}(\cdot,\overline{t})} \overline{\rho}\, d\overline{s} = - \int_{\overline{\gamma}(\cdot,\overline{t})} \overline{\rho} \left| \frac{\partial^2 \overline{\gamma}}{\partial \overline{s}^2} + \overline{\gamma}^{\perp} \right|^2 d\overline{s}, \quad (3.28)$$

where $\gamma^{\perp} = \gamma - \gamma^{\top}$*, and* $\gamma^{\top}$ *is the tangential component of the position vector.*

Proof

First we compute $\frac{\partial \overline{v}}{\partial \overline{t}}$. Arclength is given by

$$\overline{s}(p,\overline{t}) = \int_0^p \left| \frac{\partial \overline{\gamma}}{\partial q}(q,\overline{t}) \right| \, dq.$$

Differentiating,

$$\frac{\partial \overline{s}}{\partial p}(p,\overline{t}) = \left| \frac{\partial \overline{\gamma}}{\partial p}(p,\overline{t}) \right| = \overline{v}(p,\overline{t}).$$

In addition,

$$\overline{v}^2 = \left\langle \frac{\partial \overline{\gamma}}{\partial p}, \frac{\partial \overline{\gamma}}{\partial p} \right\rangle,$$

which implies that

$$2\overline{v}\frac{\partial \overline{v}}{\partial \overline{t}} = 2\left\langle \frac{\partial^2 \overline{\gamma}}{\partial \overline{t} \partial p}, \frac{\partial \overline{\gamma}}{\partial p} \right\rangle.$$

Thus,

$$\begin{aligned}
\overline{v}\frac{\partial \overline{v}}{\partial \overline{t}} &= \left\langle \frac{\partial}{\partial p}(\overline{k}\,\overline{N} + \overline{\gamma}), \frac{\partial \overline{\gamma}}{\partial p} \right\rangle \quad \text{because} \quad \frac{\partial \overline{\gamma}}{\partial \overline{t}} = \frac{\partial^2 \overline{\gamma}}{\partial \overline{s}^2} + \overline{\gamma} \\
&= \left\langle \overline{k}\frac{\partial \overline{N}}{\partial p} + \frac{\partial \overline{k}}{\partial p}\overline{N} + \frac{\partial \overline{\gamma}}{\partial p}, \frac{\partial \overline{\gamma}}{\partial p} \right\rangle \\
&= \left\langle \overline{k}\left(\overline{v}(-\overline{k}\,\overline{T})\right), \frac{\partial \overline{\gamma}}{\partial p} \right\rangle + \left| \frac{\partial \overline{\gamma}}{\partial p} \right|^2 \\
&= -\overline{k}^2 \overline{v} \left| \frac{\partial \overline{\gamma}}{\partial p} \right| + \left| \frac{\partial \overline{\gamma}}{\partial p} \right|^2 \\
&= -\overline{k}^2 \overline{v}^2 + \overline{v}^2.
\end{aligned}$$

Hence,

$$\frac{\partial \overline{v}}{\partial \overline{t}} = (-\overline{k}^2 + 1)\overline{v}.$$

Now we complete the proof of (3.28):

$$
\begin{aligned}
\frac{d}{d\bar{t}}\int_{\overline{\gamma}(\cdot,\bar{t})}\overline{\rho}\,d\overline{s} &= \frac{d}{d\bar{t}}\int_{\overline{\gamma}(\cdot,\bar{t})} e^{-\frac{1}{2}\left\langle\overline{\gamma}(p,\bar{t}),\,\overline{\gamma}(p,\bar{t})\right\rangle}\overline{v}\,dp \\
&= \int_{\overline{\gamma}(\cdot,\bar{t})}\left[\overline{\rho}(-\overline{k}^2+1)\overline{v}+\overline{\rho}(-1)\left\langle\overline{\gamma},\,\frac{\partial\overline{\gamma}}{\partial\bar{t}}\right\rangle\overline{v}\right]dp \\
&= \int_{\overline{\gamma}(\cdot,\bar{t})}\left[\overline{\rho}\left(-\left|\frac{\partial^2\overline{\gamma}}{\partial\overline{s}^2}\right|^2+1\right)-\overline{\rho}\left\langle\overline{\gamma},\,\frac{\partial^2\overline{\gamma}}{\partial\overline{s}^2}+\overline{\gamma}\right\rangle\right]d\overline{s} \\
&= \int_{\overline{\gamma}(\cdot,\bar{t})}\left[-\overline{\rho}\left(\left|\frac{\partial^2\overline{\gamma}}{\partial\overline{s}^2}+\overline{\gamma}\right|^2-2\left\langle\overline{\gamma},\,\frac{\partial^2\overline{\gamma}}{\partial\overline{s}^2}\right\rangle-|\overline{\gamma}|^2\right)+\overline{\rho}-\overline{\rho}\left(\left\langle\overline{\gamma},\,\frac{\partial^2\overline{\gamma}}{\partial\overline{s}^2}\right\rangle+|\overline{\gamma}|^2\right)\right]d\overline{s} \\
&= \int_{\overline{\gamma}(\cdot,\bar{t})}\left[-\overline{\rho}\left|\frac{\partial^2\overline{\gamma}}{\partial\overline{s}^2}+\overline{\gamma}\right|^2+\overline{\rho}+\overline{\rho}\left\langle\overline{\gamma},\,\frac{\partial^2\overline{\gamma}}{\partial\overline{s}^2}\right\rangle\right]d\overline{s} \\
&= \int_{\overline{\gamma}(\cdot,\bar{t})}\left[-\overline{\rho}\left|\frac{\partial^2\overline{\gamma}}{\partial\overline{s}^2}+\overline{\gamma}\right|^2+\overline{\rho}\frac{\partial}{\partial\overline{s}}\left\langle\overline{\gamma},\,\frac{\partial\overline{\gamma}}{\partial\overline{s}}\right\rangle\right]d\overline{s} \\
&= \int_{\overline{\gamma}(\cdot,\bar{t})}-\overline{\rho}\left|\frac{\partial^2\overline{\gamma}}{\partial\overline{s}^2}+\overline{\gamma}\right|^2d\overline{s}-\int_{\overline{\gamma}(\cdot,\bar{t})}\frac{\partial\overline{\rho}}{\partial\overline{s}}\left\langle\overline{\gamma},\,\frac{\partial\overline{\gamma}}{\partial\overline{s}}\right\rangle d\overline{s} \\
&= \int_{\overline{\gamma}(\cdot,\bar{t})}-\overline{\rho}\left(\left\langle\frac{\partial^2\overline{\gamma}}{\partial\overline{s}^2}\,\frac{\partial^2\overline{\gamma}}{\partial\overline{s}^2}\right\rangle+2\left\langle\frac{\partial^2\overline{\gamma}}{\partial\overline{s}^2},\,\overline{\gamma}\right\rangle+\langle\overline{\gamma},\,\overline{\gamma}\rangle\right)d\overline{s}-\int_{\overline{\gamma}(\cdot,\bar{t})}\overline{\rho}(-1)\left\langle\overline{\gamma},\,\frac{\partial\overline{\gamma}}{\partial\overline{s}}\right\rangle^2d\overline{s} \\
&= \int_{\overline{\gamma}(\cdot,\bar{t})}\Bigg[-\overline{\rho}\left(\left\langle\frac{\partial^2\overline{\gamma}}{\partial\overline{s}^2}\,\frac{\partial^2\overline{\gamma}}{\partial\overline{s}^2}\right\rangle+2\left\langle\frac{\partial^2\overline{\gamma}}{\partial\overline{s}^2},\,\overline{\gamma}^\perp\right\rangle+\left\langle\overline{\gamma}^\perp,\,\overline{\gamma}^\perp\right\rangle+\left\langle\overline{\gamma}^\top,\,\overline{\gamma}^\top\right\rangle\right) \\
&\qquad +\overline{\rho}\left\langle\overline{\gamma},\,\frac{\partial\overline{\gamma}}{\partial\overline{s}}\right\rangle^2\Bigg]d\overline{s} \\
&= \int_{\overline{\gamma}(\cdot,\bar{t})}\left[-\overline{\rho}\left|\frac{\partial^2\overline{\gamma}}{\partial\overline{s}^2}+\overline{\gamma}^\perp\right|^2-\overline{\rho}\left\langle\overline{\gamma},\,\frac{\partial\overline{\gamma}}{\partial\overline{s}}\right\rangle^2+\overline{\rho}\left\langle\overline{\gamma},\,\frac{\partial\overline{\gamma}}{\partial\overline{s}}\right\rangle^2\right]d\overline{s} \\
&= \int_{\overline{\gamma}(\cdot,\bar{t})}-\overline{\rho}\left|\frac{\partial^2\overline{\gamma}}{\partial\overline{s}^2}+\overline{\gamma}^\perp\right|^2d\overline{s}.
\end{aligned}
$$

$\square$

We will use theorem 3.3.1(2) to study the behavior of $\overline{\gamma}(\cdot,\bar{t})$ as $\bar{t} \to \infty$. First notice that $\overline{\gamma}(\cdot,\bar{t})$ cannot disappear at infinity. Let $\overrightarrow{0} \in \mathbb{R}^2$ be the blow-up point. Then we have

$$\begin{aligned} |\gamma(p,t)| &\leq \int_t^{\omega} |k| d\tau \\ &\leq C\sqrt{\omega - t}, \end{aligned}$$

and so,

$$|\overline{\gamma}(p,\bar{t})| \leq C.$$

Now, integrating the monotonicity formula in time gives the following lemma.

Lemma 3.3.1.

$$\int_{t_0}^{\infty} \int_{\overline{\gamma}(\cdot,\bar{t})} \bar{\rho} \left| \frac{\partial^2 \overline{\gamma}}{\partial \bar{s}^2} + \overline{\gamma}^{\perp} \right|^2 d\bar{s} d\bar{t} < \infty.$$

Corollary 3.3.1. $\forall \epsilon > 0, \quad \exists T < \infty$ *such that*

$$\int_{T}^{\infty} \int_{\overline{\gamma}(\cdot,\bar{t})} \bar{\rho} \left| \frac{\partial^2 \overline{\gamma}}{\partial \bar{s}^2} + \overline{\gamma}^{\perp} \right|^2 d\bar{s} d\bar{t} < \epsilon.$$

From the corollary above we get

$$\left| \frac{\partial^2 \overline{\gamma}}{\partial \bar{s}^2} + \overline{\gamma}^{\perp} \right| = 0.$$

That is,

$$\bar{k} = <\overline{\gamma}, -N> .$$

Hence the limit is an asymptotically self-similar homothetically shrinking solution. These are classified by Abresch and Langer [AL86], and the only embedded one is the circle. Hence if the forming singularity is type-I, then the curve converges to a round point in the C^∞ sense.

3.3.2 Type-II Singularities

We will now assume a type-II singularity is forming at time ω. Our model for this type of behavior is the formation of a cusp.

We will use the re-scaling from previous section. By [Alt91], the limit solution γ_∞ exists for all time and the curvature k satisfies $0 < k \leq 1$, and $k = 1$ at the origin at $t = 0$. Therefore, by [Ham95a], the limit is a translating soliton. It is then necessarily the graph $y = f(x,t)$ of a function where

$$\frac{\partial y}{\partial t} = \frac{\partial}{\partial x} \tan^{-1}\left(\frac{\partial y}{\partial x}\right) = 1,$$

which is solved to give the grim reaper

$$y = t + \ln(\sec x).$$

In the grim reaper, a horizontal line segment has length $L < \pi$, while if it is high enough, it encloses an arbitrarily large area A_1, while there is still an arbitrarily large area A_2 on the other side if we go out far enough. If the grim reaper is to be the limit, then the original curve comes arbitrarily close to it after translating, rotating, and dilating; all of which do not affect the constant $\overline{G}$. But then we must have $\overline{G} \to 0$, which is impossible.

Thus we have proved the following main theorem.

Main Theorem. *Let γ be a closed embedded curve evolving by curvature flow on a smooth compact Riemannian surface. If a singularity develops in finite time, then the curve converges to a round point in the C^∞ sense.*

Chapter 4

Distance Comparison Principles for Evolving Curves

Huisken [Hui98] showed that the curvature flow of an embedded curve in a plane converges smoothly to a round point by using a distance comparison principle to measure the deviation of the evolving curve from a round circle and to eliminate type-II singularities for the curvature flow.

From Chapter two, we know that when a closed curve evolves under the curvature flow in a surface, the solution remains smooth and embedded as long as its curvature remains bounded. If a singularity develops in finite time then the curve shrinks to a point. So when t is close enough to the blow-up time ω, we may assume that the curve is contained in a small neighborhood of the collapsing point on the surface. Now by theorem 2.3.5, using a local conformal diffeomorphism $\phi : U(\subseteq M) \to U' \subseteq \mathbb{R}^2$ between compact neighborhoods, we get a corresponding flow in the plane which satisfies the following equation:

$$\frac{\partial \gamma'}{\partial t} = \left(\frac{k'}{J^2} - \frac{\nabla_N J}{J^2} \right) N', \tag{4.1}$$

where $\gamma'(p,t) = \phi(\gamma(p,t))$, k' is the curvature of γ' in U', and N' is the unit normal vector.

In this chapter, we will apply Huisken's techniques to the flow (4.1) in $\mathbb{R}^2$ which corresponds to the curvature flow in a surface.

We define the extrinsic and intrinsic distance functions

$$d, l : \Gamma \times \Gamma \times [0,T] \to \mathbb{R}$$

by

$$d(p,q,t) = |\gamma(p,t) - \gamma(q,t)|_{\mathbb{R}^2},$$

and

$$l(p,q,t) = \int_p^q ds_t = s_t(q) - s_t(p),$$

where Γ is either S^1 or an interval. Notice that $0 < \frac{d}{l} \leq 1$, with equality on the diagonal of $\Gamma \times \Gamma$ or if γ is a straight line. The ratio $\frac{d}{l}$ can be considered as a measure for the straightness of an embedded curve.

4.1 Comparison between Extrinsic Distance and Intrinsic Distance

In this section we will prove our next main result, that under the parabolic flow (4.1), the ratio $\frac{d}{l}$ improves at a local minimum. This proves that embedded curves stay embedded for this parabolic flow.

Lemma B. *Let $\gamma : I \times [0,T] \to \mathbb{R}^2$ be a smooth embedded solution of the flow (4.1), where I is an interval such that l is smoothly defined on $I \times I$. Suppose $\frac{d}{l}$ attains a local minimum at (p_0, q_0) in the interior of $I \times I$ at time $t_0 \in [0,T]$. Then*

$$\frac{d}{dt}\left(\frac{d}{l}\right)(p_0, q_0, t_0) \geq 0,$$

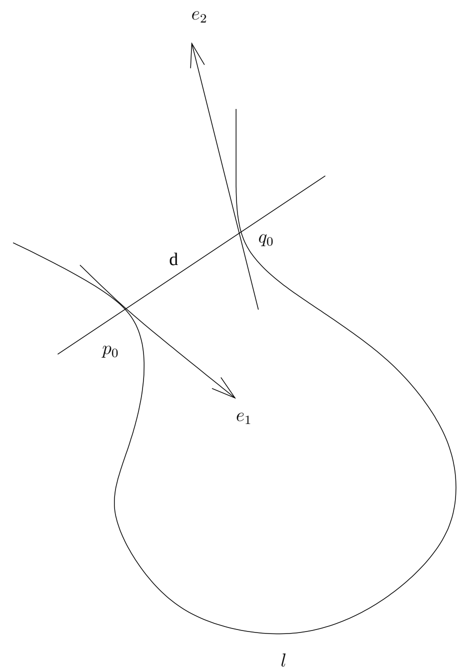
with equality if and only if γ is a straight line.

Proof

We may assume, without loss of generality, that $p_0 \neq q_0$, and $s(q_0, t_0) > s(p_0, t_0)$. Since $\frac{d}{l}$ attains a local minimum at (p_0, q_0), we have

$$\delta(\xi)\left(\frac{d}{l}\right)(p_0, q_0, t_0) = 0, \quad \text{and} \quad \delta^2(\xi)\left(\frac{d}{l}\right)(p_0, q_0, t_0) \geq 0, \tag{4.2}$$

where $\delta(\xi)$ and $\delta^2(\xi)$ denote the first and second variation with regard to the variation vector $\xi = v_1 \oplus v_2 \in T_{p_0}\gamma_{t_0} \oplus T_{q_0}\gamma_{t_0}$.



We have

$$\delta(\xi)\left(\frac{d}{l}\right)(p_0,q_0,t_0) := \left.\frac{d}{d\tau}\right|_{\tau=0}\left(\frac{d}{l}\right)(\alpha_p(\tau),\alpha_q(\tau),t_0),$$

where

$$\alpha_p(0)=\gamma(p_0,t_0) \quad \text{and} \quad \alpha_p'(0)=v_1\in T_{p_0}\gamma_{t_0},$$

$$\alpha_q(0)=\gamma(q_0,t_0) \quad \text{and} \quad \alpha_q'(0)=v_2\in T_{q_0}\gamma_{t_0}.$$

Also,

$$\delta^2(\xi)\left(\frac{d}{l}\right)(p_0,q_0,t_0) := \left.\frac{d^2}{d\tau^2}\right|_{\tau=0}\left(\frac{d}{l}\right)(\alpha_p(\tau),\alpha_q(\tau),t_0).$$

Let

$$e_1=\frac{\partial\gamma}{\partial s}(p_0,t_0), \quad e_2=\frac{\partial\gamma}{\partial s}(q_0,t_0), \quad \text{and} \quad \omega=\frac{\gamma(q_0,t_0)-\gamma(p_0,t_0)}{d(p_0,q_0,t_0)}.$$

Then using (4.2), we will show that $\langle \omega, e_1 \rangle = \langle \omega, e_2 \rangle = \frac{d}{l}$.

First, calculate $\delta(e_1 \oplus e_2)d(p_0, q_0, t_0)$:

$$\begin{aligned}\frac{d}{d\tau}d(\alpha_p(\tau), \alpha_q(\tau), t_0) &= \frac{d}{d\tau}\sqrt{\langle \alpha_p(\tau) - \alpha_q(\tau),\, \alpha_p(\tau) - \alpha_q(\tau)\rangle} \\ &= \frac{\big\langle \alpha_p'(\tau) - \alpha_q'(\tau),\, \alpha_p(\tau) - \alpha_q(\tau)\big\rangle}{\sqrt{\langle \alpha_p(\tau) - \alpha_q(\tau),\, \alpha_p(\tau) - \alpha_q(\tau)\rangle}}.\end{aligned}$$

Therefore,

$$\begin{aligned}\delta(e_1 \oplus e_2)d(p_0, q_0, t_0) &:= \left.\frac{d}{d\tau}\right|_{\tau=0} d(\alpha_p(\tau), \alpha_q(\tau), t_0) \\ &= \frac{\big\langle \alpha_p'(0) - \alpha_q'(0),\, \alpha_p(0) - \alpha_q(0)\big\rangle}{d(p_0, q_0, t_0)} \\ &= \langle e_1 - e_2, -\omega\rangle\,. \qquad (4.3)\end{aligned}$$

Assume $\xi = e_1 \oplus 0$:

Then using $\delta(e_1 \oplus 0)d(p_0, q_0, t_0) = \langle e_1, -\omega\rangle$ and

$\delta(e_1 \oplus 0)l(p_0, q_0, t_0) = \frac{d}{d\tau}|_{\tau=0}(l(p_0, q_0, t_0) - \tau) = -1$, we get

$$\begin{aligned}0 &= \delta(e_1 \oplus 0)\left(\frac{d}{l}\right)(p_0, q_0, t_0) \\ &= \frac{l(p_o, q_0, t_0)\delta(e_1 \oplus 0)d(p_0, q_0, t_0) - d(p_0, q_0, t_0)\delta(e_1 \oplus 0)l(p_0, q_0, t_0)}{l^2} \\ &= \frac{l\,\langle e_1, -\omega\rangle - d(-1)}{l^2}.\end{aligned}$$

Hence,

$$\langle \omega, e_1\rangle = \frac{d}{l}. \qquad (4.4)$$

Assume $\xi = 0 \oplus e_2$:

Then using $\delta(0 \oplus e_2)d(p_0, q_0, t_0) = \langle -e_2, -\omega \rangle$ and $\delta(0 \oplus e_2)l(p_0, q_0, t_0) = 1$, we get

$$\begin{aligned} 0 &= \delta(0 \oplus e_2)\left(\frac{d}{l}\right)(p_0, q_0, t_0) \\ &= \frac{l\langle e_2, \omega\rangle - d(1)}{l^2}, \end{aligned}$$

and hence,

$$\langle \omega, e_2 \rangle = \frac{d}{l}. \tag{4.5}$$

Then either $e_1 = e_2$ or $e_1 \neq e_2$. Notice that in the later case $e_1 + e_2$ is parallel to ω.

Case 1: $e_1 = e_2$. In this case, we choose $\xi = e_1 \oplus e_2$.

Since $\frac{d}{d\tau} l(\alpha_p(\tau), \alpha_q(\tau), t_0) = 0$, we have $\delta(\xi)(l) = 0$. Hence,

$$\begin{aligned} 0 &\leq \delta^2(\xi)\left(\frac{d}{l}\right)(p_0, q_0, t_0) = \left.\frac{d^2}{d\tau^2}\right|_{\tau=0}\left(\frac{d}{l}\right)(\alpha_p(\tau), \alpha_q(\tau), t_0) \\ &= \left.\frac{d}{d\tau}\right|_{\tau=0}\frac{l\frac{d}{d\tau}d - d\frac{d}{d\tau}l}{l^2} = \left.\frac{d}{d\tau}\right|_{\tau=0}\frac{1}{l}\frac{d}{d\tau}d \\ &= \left.\frac{d}{d\tau}\right|_{\tau=0}\left[\frac{1}{ld}\left\langle \alpha_p'(\tau) - \alpha_q'(\tau),\, \alpha_p(\tau) - \alpha_q(\tau)\right\rangle\right] \\ &= \left[\frac{ld}{(ld)^2}\left(\left\langle \alpha_p'(\tau) - \alpha_q'(\tau),\, \alpha_p'(\tau) - \alpha_q'(\tau)\right\rangle + \left\langle \alpha_p''(\tau) - \alpha_q''(\tau),\, \alpha_p(\tau) - \alpha_q(\tau)\right\rangle\right)\right. \\ &\qquad \left. + \frac{1}{(ld)^2}\left\langle \alpha_p'(\tau) - \alpha_q'(\tau),\, \alpha_p(\tau) - \alpha_q(\tau)\right\rangle \frac{d}{d\tau}(ld)\right]_{\tau=0} \\ &= \frac{1}{l}\left\langle \overrightarrow{k}(p_0, t_0) - \overrightarrow{k}(q_0, t_0),\, -\omega\right\rangle \quad \text{Since} \quad \alpha_p'(0) - \alpha_q'(0) = 0. \end{aligned}$$

Thus,

$$\left\langle \omega,\, \overrightarrow{k}(q_0, t_0) - \overrightarrow{k}(p_0, t_0)\right\rangle \geq 0. \tag{4.6}$$

Case 2: $e_1 \neq e_2$. In this case, we choose $\xi = e_1 \ominus e_2$.

Since $\frac{d}{d\tau} l(\alpha_p(\tau), \alpha_q(\tau), t_0) = \frac{d}{d\tau}(l(p_0, q_0, t_0) - 2\tau) = -2$, we have $\delta(\xi)(l) = -2$.

Thus,

$$
\begin{aligned}
0 \leq \delta^2(\xi)\left(\frac{d}{l}\right)(p_0, q_0, t_0) &= \left.\frac{d^2}{d\tau^2}\right|_{\tau=0} \left(\frac{d}{l}\right)(\alpha_p(\tau), \alpha_q(\tau), t_0) \\
&= \left.\frac{d}{d\tau}\right|_{\tau=0} \frac{l\frac{d}{d\tau}d - d\frac{d}{d\tau}l}{l^2} \\
&= \left.\frac{d}{d\tau}\right|_{\tau=0} \left[\frac{1}{ld}\left\langle \alpha_p'(\tau) - \alpha_q'(\tau),\, \alpha_p(\tau) - \alpha_q(\tau)\right\rangle + \frac{2d}{l^2}\right] \\
&= \Bigg[\frac{ld}{(ld)^2}\left(\left\langle \alpha_p'(\tau) - \alpha_q'(\tau),\, \alpha_p'(\tau) - \alpha_q'(\tau)\right\rangle + \left\langle \alpha_p''(\tau) - \alpha_q''(\tau),\, \alpha_p(\tau) - \alpha_q(\tau)\right\rangle\right) \\
&\quad - \frac{1}{(ld)^2}\left\langle \alpha_p'(\tau) - \alpha_q'(\tau),\, \alpha_p(\tau) - \alpha_q(\tau)\right\rangle \left(l\frac{d}{d\tau}d + d\frac{d}{d\tau}l\right) + \left(\frac{2l^2\frac{d}{d\tau}d - 2d.2l.\frac{d}{d\tau}l}{l^4}\right)\Bigg]_{\tau=0} \\
&= \frac{1}{l}\left\langle \overrightarrow{k}(p_0, t_0) - \overrightarrow{k}(q_0, t_0),\, -\omega\right\rangle + \frac{1}{ld}|e_1 + e_2|^2 + \frac{1}{l^2 d}\left\langle e_1 + e_2,\, \omega\right\rangle \\
&\qquad (l\left\langle e_1 + e_2, -\omega\right\rangle - 2d) + \frac{2}{l^2}\left\langle e_1 + e_2,\, -\omega\right\rangle - \frac{4d}{l^3}(-2) \\
&= \frac{1}{l}\left\langle \omega,\, \overrightarrow{k}(q_0, t_0) - \overrightarrow{k}(p_0, t_0)\right\rangle + \frac{1}{ld}|e_1 + e_2|^2 - \frac{1}{ld}\left\langle e_1 + e_2,\, \omega\right\rangle^2 \\
&\qquad - \frac{4}{l^2}\left\langle e_1 + e_2,\, \omega\right\rangle + \frac{8d}{l^3} \\
&= \frac{1}{l}\left\langle \omega,\, \overrightarrow{k}(q_0, t_0) - \overrightarrow{k}(p_0, t_0)\right\rangle.
\end{aligned}
$$

The last line follows from $\langle \omega,\ e_1 + e_2\rangle = \frac{2d}{l}$, which implies that $-\frac{4}{l^2}\langle e_1 + e_2,\, \omega\rangle = -\frac{8d}{l^3}$. Also $\omega \parallel e_1 + e_2$ gives $\langle e_1 + e_2,\, \omega\rangle^2 = |e_1 + e_2|^2$, and so $\frac{1}{ld}\langle e_1 + e_2,\, \omega\rangle^2 = \frac{1}{ld}|e_1 + e_2|^2$.

Hence,

$$\left\langle \omega,\, \overrightarrow{k}(q_0, t_0) - \overrightarrow{k}(p_0, t_0)\right\rangle \geq 0. \tag{4.7}$$

We now use the evolution equation (4.1) to compute $\frac{d}{dt}\left(\frac{d}{l}\right)(p_0, q_0, t_0)$.

$$\frac{d}{dt}\left(\frac{d}{l}\right)(p_0, q_0, t_0) = \frac{l\frac{d}{dt}d - d\frac{d}{dt}l}{l^2} = \frac{1}{l}\frac{d}{dt}d - \frac{d}{l^2}\frac{d}{dt}l.$$

we have

$$\begin{aligned}
\frac{d}{dt}d(p_o, q_0, t_0) &= \frac{d}{dt}\sqrt{\left\langle \gamma(p_0, t_0) - \gamma_(q_0, t_0), \gamma(p_0, t_0) - \gamma_(q_0, t_0) \right\rangle} \\
&\frac{\left\langle \frac{\partial \gamma}{\partial t}(p_0, t_0) - \frac{\partial \gamma}{\partial t}(q_0, t_0), \gamma(p_0, t_0) - \gamma(q_0, t_0) \right\rangle}{d(p_0, q_0, t_0)} \\
&= \left\langle \frac{\partial \gamma}{\partial t}(p_0, t_0) - \frac{\partial \gamma}{\partial t}(q_0, t_0), -\omega \right\rangle \\
&= \left\langle \left(\frac{k}{J^2} - \frac{\nabla_N J}{J^2}\right) N(p_0, t_0) - \left(\frac{k}{J^2} - \frac{\nabla_N J}{J^2}\right) N(q_0, t_0), -\omega \right\rangle,
\end{aligned}$$

and

$$\frac{d}{dt}l = \frac{d}{dt}\int_p^q ds_t = -\int_p^q k\left(\frac{k}{J^2} - \frac{\nabla_N J}{J^2}\right) ds_t.$$

Since $\langle \omega, e_1 \rangle = \langle \omega, e_2 \rangle = \frac{d}{l}$, let $\alpha = \angle(\omega, e_1) = \angle(\omega, e_2)$ with $0 < \alpha < \pi/2$. Then $\langle \omega,\ N(q_0, t_0) \rangle = \sin\alpha$ and $\langle \omega,\ N(p_0, t_0) \rangle = -\sin\alpha$. Since $\left\langle \omega,\ \overrightarrow{k}(q_0, t_0) - \overrightarrow{k}(p_0, t_0) \right\rangle \geq 0$, we have $k(p_0, t_0) + k(q_0, t_0) \geq 0$. Therefore,

$$\frac{d}{dt}d(p_o, q_0, t_0) = \left(\frac{k}{J^2}(p_0, t_0) + \frac{k}{J^2}(q_0, t_0)\right)\sin\alpha - \left(\frac{\nabla_N J}{J^2}(p_0, t_0) + \frac{\nabla_N J}{J^2}(q_0, t_0)\right)\sin\alpha.$$

Hence,

$$
\begin{aligned}
&\frac{d}{dt}\left(\frac{d}{l}\right)(p_0, q_0, t_0) \\
&= \frac{1}{l}\frac{d}{dt}d - \frac{d}{l^2}\frac{d}{dt}l \\
&= \frac{1}{l}\left(\left(\frac{k}{J^2}(p_0, t_0) + \frac{k}{J^2}(q_0, t_0)\right) - \left(\frac{\nabla_N J}{J^2}(p_0, t_0) + \frac{\nabla_N J}{J^2}(q_0, t_0)\right)\right)\sin\alpha \\
&\qquad + \frac{d}{l^2}\int_p^q k\left(\frac{k}{J^2} - \frac{\nabla_N J}{J^2}\right) ds_t \\
&\geq \frac{1}{l}\left[(k(p_0, t_0) + k(q_0, t_0))C_1 - 2C_2\right]\sin\alpha + \frac{d}{l^2}C_1\int_p^q k^2\, ds - \frac{d}{l^2}C_2\int_p^q k\, ds \\
&= \left[\frac{\sin\alpha}{l}(k(p_0, t_0) + k(q_0, t_0)) + \frac{d}{l^2}\int_p^q k^2\, ds\right]C_1 - \left[\frac{2\sin\alpha}{l} + \frac{d}{l^2}\int_p^q k\, ds\right]C_2,
\end{aligned}
$$

where

$$
C_1 = \min_U\left(\frac{1}{J^2}\right), \quad C_2 = \max_U\left(\frac{\nabla_N J}{J^2}\right).
$$

If t_0 is close enough to the blow-up time ω, we can make C_1 approach 1 and C_2 approach 0. Since $\int_p^q k^2\, ds > 0$, and $\frac{2\sin\alpha}{l} + \frac{d}{l^2}\int_p^q k\, ds$ is bounded, we have $\frac{d}{dt}\left(\frac{d}{l}\right)(p_0, q_0, t_0) > 0$. □

4.2 Deviation of the Evolving Curve from a Circle

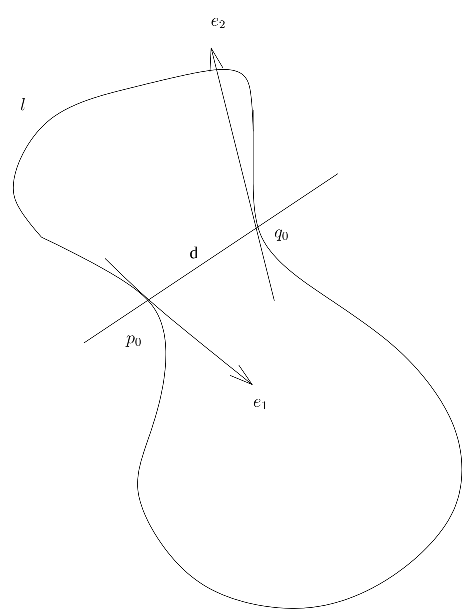
Now let $\gamma : S^1 \times [0, T] \to \mathbb{R}^2$ be a closed smooth embedded curve moving by the flow (4.1). Let $L(t)$ be the total length of the curve. The intrinsic distance function l is now only smoothly defined for $0 \leq l < \frac{L}{2}$. We define the smooth function

$$\psi : S^1 \times S^1 \times [0,T] \to \mathbb{R}$$

by

$$\psi : (p,q,t) := \frac{L(t)}{\pi} \sin\left(\frac{l(p,q,t)\pi}{L(t)}\right).$$

So the isoperimetric ratio $\frac{d}{\psi} = \frac{d}{l}\left(\frac{\frac{l\pi}{L}}{\sin(\frac{l\pi}{L})}\right) \to 1$ on the diagonal of $S^1 \times S^1$ and $\frac{d}{\psi} \equiv 1$ on any circle. We now prove our last main result: the ratio $\frac{d}{\psi}$ improves at a local minimum under the parabolic flow (4.1). Therefore, it plays the role of an improving isoperimetric ratio that measures the deviation of the evolving curve from a circle.



Lemma C. *Let $\gamma : S^1 \times [0,T] \to \mathbb{R}^2$ be a smooth embedded solution of the flow (4.1). Suppose $\frac{d}{\psi}$ attains a local minimum $(\frac{d}{\psi})(p_0, q_0, t_0) < 1$ at some* *point* $(p_0, q_0) \in S^1 \times S^1$ *at time* $t_0 \in [0, T]$*. Then*

$$\frac{d}{dt}\left(\frac{d}{\psi}\right)(p_0, q_0, t_0) \geq 0,$$

with equality if and only if $\frac{d}{\psi} \equiv 1$ *or* $\gamma(S^1, \cdot)$ *is a circle.*

Proof

We may assume, without loss of generality, that $0 = s(p_0, t_0) < s(q_0, t_0) < \frac{L(t_0)}{2}$, such that $l(p_0, q_0, t_0) = s(q_0, t_0) - s(p_0, t_0)$. Since $\frac{d}{\psi}$ attains a local minimum at (p_0, q_0), we have

$$\delta(\xi)\left(\frac{d}{\psi}\right)(p_0, q_0, t_0) = 0, \quad \text{and} \quad \delta^2(\xi)\left(\frac{d}{\psi}\right)(p_0, q_0, t_0) \geq 0, \tag{4.8}$$

where $\delta(\xi)$ and $\delta^2(\xi)$ denote the first and second variation with regard to the variation vector $\xi = v_1 \oplus v_2 \in T_{p_0}\gamma_{t_0} \oplus T_{q_0}\gamma_{t_0}$. Let

$$e_1 = \frac{\partial \gamma}{\partial s}(p_0, t_0), \quad e_2 = \frac{\partial \gamma}{\partial s}(q_0, t_0), \quad \text{and} \quad \omega = \frac{\gamma(q_0, t_0) - \gamma(p_0, t_0)}{d(p_0, q_0, t_0)}.$$

Then using (4.8), we first show that $\langle \omega, e_1 \rangle = \langle \omega, e_2 \rangle = \frac{d}{\psi}\cos(\frac{l\pi}{L})$.

Now from (4.3), we have

$$\delta(e_1 \oplus e_2) d(p_0, q_0, t_0) = \langle e_1 - e_2, -\omega \rangle,$$

and

$$\begin{aligned} \frac{d}{d\tau}\psi(\alpha_p(\tau), \alpha_q(\tau), t_0) &= \frac{L}{\pi}\cos\left(\frac{l\pi}{L}\right)\frac{\pi}{L}\frac{d}{d\tau}(l) \\ &= \cos\left(\frac{l\pi}{L}\right)\frac{d}{d\tau}(l). \end{aligned}$$

Assume $\xi = e_1 \oplus 0$: Then

$$\begin{aligned} 0 &= \delta(e_1 \oplus 0)\left(\frac{d}{\psi}\right)(p_0, q_0, t_0) \\ &= \frac{\psi(p_o, q_0, t_0)\delta(e_1 \oplus 0)d(p_0, q_0, t_0) - d(p_0, q_0, t_0)\delta(e_1 \oplus 0)\psi(p_0, q_0, t_0)}{l^2} \\ &= \frac{\psi \langle e_1, -\omega\rangle - d\cos(\frac{l\pi}{L})(-1)}{\psi^2}, \end{aligned}$$

and hence,

$$\langle \omega, e_1 \rangle = \frac{d}{\psi}\cos\left(\frac{l\pi}{L}\right). \tag{4.9}$$

Assume $\xi = 0 \oplus e_2$:

$$\begin{aligned} 0 &= \delta(0 \oplus e_2)\left(\frac{d}{\psi}\right)(p_0, q_0, t_0) \\ &= \frac{\psi \langle e_2, \omega\rangle - d\cos(\frac{l\pi}{L})(1)}{l^2}, \end{aligned}$$

and hence,

$$\langle \omega, e_2 \rangle = \frac{d}{\psi}\cos\left(\frac{l\pi}{L}\right). \tag{4.10}$$

Then either $e_1 = e_2$ or $e_1 \neq e_2$. Notice that, in the later case, $e_1 + e_2$ is parallel to ω.

Case 1: $e_1 = e_2$. In this case, we choose $\xi = e_1 \oplus e_2$.

Since $\frac{d}{d\tau}\psi(\alpha_p(\tau), \alpha_q(\tau), t_0) = 0$ we have $\delta(\xi)(\psi) = 0$. Then

$$
\begin{aligned}
0 \le \delta^2(\xi)\left(\frac{d}{\psi}\right)(p_0, q_0, t_0) &= \left.\frac{d^2}{d\tau^2}\right|_{\tau=0}\left(\frac{d}{\psi}\right)(\alpha_p(\tau), \alpha_q(\tau), t_0) \\
&= \left.\frac{d}{d\tau}\right|_{\tau=0}\frac{\psi\frac{d}{d\tau}d - d\frac{d}{d\tau}\psi}{\psi^2} = \left.\frac{d}{d\tau}\right|_{\tau=0}\frac{1}{\psi}\frac{d}{d\tau}d \\
&= \left.\frac{d}{d\tau}\right|_{\tau=0}\left[\frac{1}{\psi d}\left\langle \alpha_p'(\tau) - \alpha_q'(\tau),\, \alpha_p(\tau) - \alpha_q(\tau)\right\rangle\right] \\
&= \left[\frac{\psi d}{(\psi d)^2}\left(\left\langle \alpha_p'(\tau) - \alpha_q'(\tau),\, \alpha_p'(\tau) - \alpha_q'(\tau)\right\rangle + \left\langle \alpha_p''(\tau) - \alpha_q''(\tau),\, \alpha_p(\tau) - \alpha_q(\tau)\right\rangle\right)\right. \\
&\qquad \left. + \frac{1}{(\psi d)^2}\left\langle \alpha_p'(\tau) - \alpha_q'(\tau),\, \alpha_p(\tau) - \alpha_q(\tau)\right\rangle \frac{d}{d\tau}(\psi d)\right]_{\tau=0} \\
&= \frac{1}{\psi}\left\langle \overrightarrow{k}(p_0, t_0) - \overrightarrow{k}(q_0, t_0),\, -\omega\right\rangle \quad \text{Since} \quad \alpha_p'(0) - \alpha_q'(0) = 0.
\end{aligned}
$$

Thus,

$$
\left\langle \omega,\, \overrightarrow{k}(q_0, t_0) - \overrightarrow{k}(p_0, t_0)\right\rangle \ge 0. \tag{4.11}
$$

Case 2: $e_1 \neq e_2$. In this case, we choose $\xi = e_1 \ominus e_2$.

Since $\frac{d}{d\tau}\psi(\alpha_p(\tau),\alpha_q(\tau),t_0) = \cos(\frac{l\pi}{L})\frac{d}{d\tau}(l(p_0,q_0,t_0)-2\tau) = \cos(\frac{l\pi}{L})(-2)$, we have $\delta(\xi)(\psi) = -2\cos(\frac{l\pi}{L})$. Then,

$$
\begin{aligned}
0 &\leq \delta^2(\xi)\left(\frac{d}{\psi}\right)(p_0,q_0,t_0) = \left.\frac{d^2}{d\tau^2}\right|_{\tau=0}\left(\frac{d}{\psi}\right)(\alpha_p(\tau),\alpha_q(\tau),t_0)\\
&= \left.\frac{d}{d\tau}\right|_{\tau=0} \frac{\psi\frac{d}{d\tau}d - d\frac{d}{d\tau}\psi}{\psi^2}\\
&= \left.\frac{d}{d\tau}\right|_{\tau=0}\left[\frac{1}{\psi d}\left\langle \alpha_p'(\tau)-\alpha_q'(\tau),\, \alpha_p(\tau)-\alpha_q(\tau)\right\rangle + \frac{2d}{\psi^2}\cos\left(\frac{l\pi}{L}\right)\right]\\
&= \Bigg[\frac{\psi d}{(\psi d)^2}\left(\left\langle \alpha_p'(\tau)-\alpha_q'(\tau),\, \alpha_p'(\tau)-\alpha_q'(\tau)\right\rangle + \left\langle \alpha_p''(\tau)-\alpha_q''(\tau),\, \alpha_p(\tau)-\alpha_q(\tau)\right\rangle\right)\\
&\quad -\frac{1}{(\psi d)^2}\left\langle \alpha_p'(\tau)-\alpha_q'(\tau),\, \alpha_p(\tau)-\alpha_q(\tau)\right\rangle\left(\psi\frac{d}{d\tau}d + d\frac{d}{d\tau}\psi\right)\\
&\quad +\frac{2d}{\psi^2}\left(-\sin\left(\frac{l\pi}{L}\right)\left(\frac{\pi}{L}\right)(-2)\right) + \cos\left(\frac{l\pi}{L}\right)\left(\frac{2\psi^2\frac{d}{d\tau}d - 2d.2\psi.\frac{d}{d\tau}\psi}{\psi^4}\right)\Bigg]_{\tau=0}\\
&= \frac{1}{\psi}\left\langle \overrightarrow{k}(p_0,t_0) - \overrightarrow{k}(q_0,t_0),\, -\omega\right\rangle + \frac{1}{\psi d}|e_1+e_2|^2 + \frac{1}{\psi^2 d}\left\langle e_1+e_2,\, \omega\right\rangle\\
&\qquad \left(\psi\left\langle e_1+e_2, -\omega\right\rangle - 2d\cos\left(\frac{l\pi}{L}\right)\right) + \frac{4d\pi}{\psi^2 L}\sin\left(\frac{l\pi}{L}\right) + \frac{2}{\psi^2}\cos\left(\frac{l\pi}{L}\right)\left\langle e_1+e_2,\, -\omega\right\rangle\\
&\quad - \frac{4d}{\psi^3}(-2)\cos^2\left(\frac{l\pi}{L}\right)\\
&= \frac{1}{\psi}\left\langle \omega,\, \overrightarrow{k}(q_0,t_0) - \overrightarrow{k}(p_0,t_0)\right\rangle + \frac{1}{\psi d}|e_1+e_2|^2 - \frac{1}{\psi d}\left\langle e_1+e_2,\, \omega\right\rangle^2\\
&\qquad - \frac{4}{\psi^2}\cos\left(\frac{l\pi}{L}\right)\left\langle e_1+e_2,\, \omega\right\rangle + \frac{4d\pi^2}{\psi^2L^2}\left(\frac{L}{\pi}\sin\left(\frac{l\pi}{L}\right)\right) + \frac{8d}{\psi^3}\cos^2\left(\frac{l\pi}{L}\right)\\
&= \frac{1}{\psi}\left\langle \omega,\, \overrightarrow{k}(q_0,t_0) - \overrightarrow{k}(p_0,t_0)\right\rangle + \frac{4\pi^2 d}{L^2\psi}.
\end{aligned}
$$

The last line follows from $\langle \omega,\, e_1+e_2\rangle = \frac{2d}{\psi}\cos\left(\frac{l\pi}{L}\right)$, which implies that $-\frac{4}{\psi^2}\cos\left(\frac{l\pi}{L}\right)\langle e_1+e_2,\, \omega\rangle = -\frac{8d}{\psi^3}\cos^2\left(\frac{l\pi}{L}\right)$. Then $\omega \parallel e_1+e_2$ gives $\langle e_1+e_2,\, \omega\rangle^2 = |e_1+e_2|^2$, and so $\frac{1}{\psi d}\langle e_1+e_2,\, \omega\rangle^2 = \frac{1}{\psi d}|e_1+e_2|^2$.

Hence,

$$\frac{1}{\psi}\left\langle \omega,\ \overrightarrow{k}(q_0,t_0)-\overrightarrow{k}(p_0,t_0)\right\rangle + \frac{4\pi^2 d}{L^2\psi}\ \geq 0. \tag{4.12}$$

We now use the evolution equation (4.1) to compute $\frac{d}{dt}\left(\frac{d}{\psi}\right)(p_0,q_0,t_0)$.

$$\frac{d}{dt}\left(\frac{d}{\psi}\right)(p_0,q_0,t_0) = \frac{\psi\frac{d}{dt}d - d\frac{d}{dt}\psi}{\psi^2} = \frac{1}{\psi}\frac{d}{dt}d - \frac{d}{\psi^2}\frac{d}{dt}\psi.$$

We have

$$\begin{aligned}
\frac{d}{dt}d(p_o,q_0,t_0) &= \frac{d}{dt}\sqrt{\langle\gamma(p_0,t_0)-\gamma_(q_0,t_0),\gamma(p_0,t_0)-\gamma_(q_0,t_0)\rangle} \\
&\frac{\langle\frac{\partial\gamma}{\partial t}(p_0,t_0)-\frac{\partial\gamma}{\partial t}(q_0,t_0),\gamma(p_0,t_0)-\gamma(q_0,t_0)\rangle}{d(p_0,q_0,t_0)} \\
&= \left\langle \frac{\partial\gamma}{\partial t}(p_0,t_0)-\frac{\partial\gamma}{\partial t}(q_0,t_0),-\omega\right\rangle \\
&= \left\langle \left(\frac{k}{J^2}-\frac{\nabla_N J}{J^2}\right)N(p_0,t_0)-\left(\frac{k}{J^2}-\frac{\nabla_N J}{J^2}\right)N(q_0,t_0),-\omega\right\rangle,
\end{aligned}$$

and

$$\begin{aligned}
\frac{d\psi}{dt} &= \frac{L}{\pi}\cos\left(\frac{l\pi}{L}\right)\pi\frac{d}{dt}\left(\frac{l}{L}\right)+\frac{dL}{dt}\frac{1}{\pi}\sin\left(\frac{l\pi}{L}\right) \\
&= L\ \cos\left(\frac{l\pi}{L}\right)\left(\frac{L\frac{d}{dt}l-l\frac{d}{dt}L}{L^2}\right)+\frac{1}{\pi}\sin\left(\frac{l\pi}{L}\right)\frac{dL}{dt} \\
&= \cos\left(\frac{l\pi}{L}\right)\frac{d(l)}{dt}+\left(\frac{1}{\pi}\sin\left(\frac{l\pi}{L}\right)-\frac{l}{L}\cos\left(\frac{l\pi}{L}\right)\right)\frac{dL}{dt} \\
&= -\cos\left(\frac{l\pi}{L}\right)\int_p^q k\left(\frac{k}{J^2}-\frac{\nabla_N J}{J^2}\right)\,ds_t \\
&\quad -\left(\frac{1}{\pi}\sin\left(\frac{l\pi}{L}\right)-\frac{l}{L}\cos\left(\frac{l\pi}{L}\right)\right)\int_{S^1}k\left(\frac{k}{J^2}-\frac{\nabla_N J}{J^2}\right)\,ds_t.
\end{aligned}$$

Since $\langle\omega,e_1\rangle=\langle\omega,e_2\rangle=\frac{d}{l}\cos(\frac{l\pi}{L})$, let $\alpha=\angle(\omega,e_1)=\angle(\omega,e_2)$ with $0<\alpha<\pi/2$.

Then $\langle\omega,\ N(q_0,t_0)\rangle=\sin\alpha$, and $\langle\omega,\ N(p_0,t_0)\rangle=-\sin\alpha$. Therefore we have

$$
\begin{aligned}
&\frac{d}{dt}\left(\frac{d}{\psi}\right)(p_0, q_0, t_0) \\
&= \frac{1}{\psi}\frac{d}{dt}d - \frac{d}{\psi^2}\frac{d}{dt}\psi \\
&= \frac{1}{\psi}\left(\left(\frac{k}{J^2}(p_0, t_0) + \frac{k}{J^2}(q_0, t_0)\right) - \left(\frac{\nabla_N J}{J^2}(p_0, t_0) + \frac{\nabla_N J}{J^2}(q_0, t_0)\right)\right)\sin\alpha \\
&\quad\; + \frac{d}{\psi^2}\cos\left(\frac{l\pi}{L}\right)\int_p^q k\left(\frac{k}{J^2} - \frac{\nabla_N J}{J^2}\right) ds_t \\
&\quad\; + \frac{d}{\psi^2}\left(\frac{1}{\pi}\sin\left(\frac{l\pi}{L}\right) - \frac{l}{L}\cos\left(\frac{l\pi}{L}\right)\right)\int_{S^1} k\left(\frac{k}{J^2} - \frac{\nabla_N J}{J^2}\right) ds_t \\
&\geq \frac{1}{\psi}[(k(p_0, t_0) + k(q_0, t_0))C_1 - 2C_2]\sin\alpha \\
&\quad\; + \frac{d}{\psi^2}\cos\left(\frac{l\pi}{L}\right)C_1\int_p^q k^2\, ds - \frac{d}{\psi^2}\cos\left(\frac{l\pi}{L}\right)C_2\int_p^q k\, ds \\
&\quad\; + \frac{d}{\psi L}\left(1 - \frac{l}{\psi}\cos\left(\frac{l\pi}{L}\right)\right)\left[C_1\int_{S^1} k^2\, ds - C_2\int_{S^1} k\, ds\right] \\
&= \left[\frac{\sin\alpha}{\psi}(k(p_0, t_0) + k(q_0, t_0)) + \frac{d}{\psi^2}\cos\left(\frac{l\pi}{L}\right)\int_p^q k^2\, ds\right. \\
&\quad \left. + \frac{d}{\psi L}\left(1 - \frac{l}{\psi}\cos\left(\frac{l\pi}{L}\right)\right)\int_{S^1} k^2\, ds\right] C_1 \\
&\quad - \left[\frac{2\sin\alpha}{\psi} + \frac{d}{\psi^2}\cos\left(\frac{l\pi}{L}\right)\int_p^q k\, ds + \frac{d}{\psi L}\left(1 - \frac{l}{\psi}\cos\left(\frac{l\pi}{L}\right)\right)\int_{S^1} k\, ds\right] C_2,
\end{aligned}
$$

where

$$
C_1 = \min_U\left(\frac{1}{J^2}\right), \quad C_2 = \max_U\left(\frac{\nabla_N J}{J^2}\right).
$$

Since

$$
\frac{l}{\psi}\cos\left(\frac{l\pi}{L}\right) = \frac{\frac{l\pi}{L}}{\tan(\frac{l\pi}{L})} < 1,
$$

we have

$$1-\frac{l}{\psi}\cos\left(\frac{l\pi}{L}\right)>0.$$

If t_0 is close enough to the blow-up time ω, we can make C_1 approach 1 and C_2 approach 0. We also have $\int k^2\,ds>0$. We now consider case 1 and case 2 separately to show $\frac{d}{dt}\left(\frac{d}{\psi}\right)(p_0,q_0,t_0)>0$.

Case 1: $e_1=e_2$. Since $\left\langle \omega,\ \overrightarrow{k}(q_0,t_0)-\overrightarrow{k}(p_0,t_0)\right\rangle \geq 0$, we have $k(p_0,t_0)+k(q_0,t_0)\geq 0$. Hence, $\frac{d}{dt}\left(\frac{d}{\psi}\right)(p_0,q_0,t_0)>0$.

Case 2: $e_1\neq e_2$. Since $\frac{1}{\psi}\left\langle \omega,\ \overrightarrow{k}(q_0,t_0)-\overrightarrow{k}(p_0,t_0)\right\rangle+\frac{4\pi^2 d}{L^2\psi}\ \geq 0$, we have $\frac{\sin\alpha}{\psi}(k(p_0,t_0)+k(q_0,t_0))\geq -\frac{4\pi^2 d}{L^2\psi}$.

<u>Claim:</u>

$$\frac{d}{\psi L}\left(1-\frac{l}{\psi}\cos\left(\frac{l\pi}{L}\right)\right)\int_{S^1}k^2\,ds-\frac{4\pi^2 d}{L^2\psi}\geq -\frac{4\pi^2 dl}{\psi^2L^2}\cos\left(\frac{l\pi}{L}\right).$$

Using the claim, we have

$$
\begin{aligned}
&\frac{d}{dt}\left(\frac{d}{\psi}\right)(p_0, q_0, t_0) \\
&\geq \left[-\frac{4\pi^2 d}{L^2\psi} + \frac{d}{\psi^2}\cos\left(\frac{l\pi}{L}\right)\int_p^q k^2\, ds \right.\\
&\quad \left. + \frac{d}{\psi L}\left(1 - \frac{l}{\psi}\cos\left(\frac{l\pi}{L}\right)\right)\int_{S^1} k^2\, ds\right] C_1 \\
&\quad - \left[\frac{2\sin\alpha}{\psi} + \frac{d}{\psi^2}\cos\left(\frac{l\pi}{L}\right)\int_p^q k\, ds + \frac{d}{\psi L}\left(1 - \frac{l}{\psi}\cos\left(\frac{l\pi}{L}\right)\right)\int_{S^1} k\, ds\right] C_2 \\
&\geq \left[\frac{d}{\psi^2}\cos\left(\frac{l\pi}{L}\right)\int_p^q k^2\, ds - \frac{4\pi^2 dl}{\psi^2 L^2}\cos\left(\frac{l\pi}{L}\right)\right] C_1 \\
&\quad - \left[\frac{2\sin\alpha}{\psi} + \frac{d}{\psi^2}\cos\left(\frac{l\pi}{L}\right)\int_p^q k\, ds + \frac{d}{\psi L}\left(1 - \frac{l}{\psi}\cos\left(\frac{l\pi}{L}\right)\right)\int_{S^1} k\, ds\right] C_2 \\
&\geq \left[\frac{d}{\psi^2 l}\cos\left(\frac{l\pi}{L}\right)\left(l\int_p^q k^2\, ds - \frac{4\pi^2 l^2}{L^2}\right)\right] C_1 \\
&\quad - \left[\frac{2\sin\alpha}{\psi} + \frac{d}{\psi^2}\cos\left(\frac{l\pi}{L}\right)\int_p^q k\, ds + \frac{d}{\psi L}\left(1 - \frac{l}{\psi}\cos\left(\frac{l\pi}{L}\right)\right)\int_{S^1} k\, ds\right] C_2.
\end{aligned}
$$

By the Hölder inequality,

$$
l\int_p^q k^2\, ds \geq \left(\int_p^q |k|\, ds\right)^2 \geq 4\alpha^2 > \frac{4\pi^2 l^2}{L^2}.
$$

The last inequality is true since $\cos\alpha = \langle \omega, e_1\rangle = \langle \omega, e_2\rangle = \frac{d}{\psi}\cos(\frac{l\pi}{L}) \Rightarrow$ $\cos\alpha < \cos(\frac{l\pi}{L}) \Rightarrow \alpha > \frac{l\pi}{L}$. Hence, $\frac{d}{dt}\left(\frac{d}{\psi}\right)(p_0, q_0, t_0) > 0$.

We now prove the claim:

$$
\begin{aligned}
\frac{d}{\psi L}\left(1-\frac{l}{\psi}\cos\left(\frac{l\pi}{L}\right)\right)\int_{S^1} k^2\,ds - \frac{4\pi^2 d}{L^2\psi} &\geq \frac{d}{\psi L}\left(1-\frac{l}{\psi}\cos\left(\frac{l\pi}{L}\right)\right)\left(\frac{4\pi^2}{l}\right) - \frac{4\pi^2 d}{L^2\psi} \\
&= \frac{4\pi^2 dl}{\psi^2 L^2}\left[\frac{\psi L}{l^2}\left(1-\frac{l}{\psi}\cos\left(\frac{l\pi}{L}\right)\right) - \frac{\psi}{l}\right] \\
&= \frac{4\pi^2 dl}{\psi^2 L^2}\left[\frac{\psi}{l}\left(\frac{L}{l}-1\right) - \frac{L}{l}\cos\left(\frac{l\pi}{L}\right)\right] \\
&> \frac{4\pi^2 dl}{\psi^2 L^2}\left[\frac{\psi}{l} - \frac{L}{l}\cos\left(\frac{l\pi}{L}\right)\right] \\
&= \frac{4\pi^2 dl}{\psi^2 L^2}\left[\frac{\sin(\frac{l\pi}{L})}{(\frac{l\pi}{L})} - \frac{L}{l}\cos\left(\frac{l\pi}{L}\right)\right] \\
&\geq \frac{4\pi^2 dl}{\psi^2 L^2}\left[\cos\left(\frac{l\pi}{L}\right) - \frac{L}{l}\cos\left(\frac{l\pi}{L}\right)\right] \\
&= -\frac{4\pi^2 dl}{\psi^2 L^2}\cos\left(\frac{l\pi}{L}\right)\left(\frac{L}{l}-1\right) \\
&= -\frac{4\pi^2 dl}{\psi^2 L^2}\cos\left(\frac{l\pi}{L}\right).
\end{aligned}
$$

□

The distance comparison principles thus established immediately rule out slowly forming (type-II) singularities for the flow, where the ratios estimated above tend to zero. Thus, using only the known classification of possible singularities, we have again proved the main theorem using distance comparison principles.

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